Only the first problem below is to be turned in.

1. Show that if \( \mu_n \Rightarrow \mu \) and \( \nu_n \Rightarrow \nu \), then \( \mu_n \ast \nu_n \Rightarrow \mu \ast \nu \).

2. (a) Let \( \{X_n, n \geq 1\} \) be independent random variables with respective characteristic functions \( \phi_n \), and let \( S_n = \sum_{k=1}^{n} X_k \). Show that if \( S_n \Rightarrow S_\infty \), then \( \prod_{n=1}^{\infty} \phi_n(t) \) converges in the sense of infinite product for each \( t \) and is the characteristic function of \( S_\infty \).

(b) Interpret and give a probabilist proof of the trigonometric identity

\[
\frac{\sin t}{t} = \prod_{n=1}^{\infty} \cos(t/2^n).
\]

Hint: Consider the probability space \((\Omega, \mathcal{F}, P) = ((0, 1], \mathcal{B}, \lambda)\), and let \( \delta_n(\omega) \) be the \( n \)th digit in the nonterminating dyadic expansion of \( \omega \in (0, 1] \), i.e.,

\[
\omega = \sum_{n=1}^{\infty} \delta_n(\omega)2^{-n}.
\]

Then the identity function \( X(\omega) = \omega \) is uniformly distributed on the unit interval, \( \{\delta_n, n \geq 1\} \) are independent Bernoulli \( (p = 1/2) \) random variables, and of course

\[
X(\omega) = \sum_{n=1}^{\infty} \delta_n(\omega)2^{-n} \quad \forall \omega \in \Omega.
\]

Now consider

\[
2X - 1 = 2 \sum_{n=1}^{\infty} \delta_n 2^{-n} = \sum_{n=1}^{\infty} (2\delta_n - 1)2^{-n}.
\]