Turn in the first two problems. The remaining problems are optional problems that you may wish to try, but they will not be collected.

1. Suppose that $E[X^2] = 1$ and $E[|X|] \geq a > 0$. Show that $P[|X| \geq \lambda a] \geq (1 - \lambda)^2 a^2$ for $0 \leq \lambda \leq 1$.

2. If $X_i \geq 0$, then
   
   $$E\left[\left(\sum_{i=1}^{n} X_i\right)^p\right] \leq \sum_{i=1}^{n} E[X_i^p]$$
   
   according as $p \leq 1$ or $p \geq 1$.

3. It was shown in class that for any $p > 0$,
   
   $E[|X + Y|^p] \leq 2^p (E[|X|^p] + E[|Y|^p])$.

   Refine this inequality by showing that:

   (a) if $p > 1$, the factor $2^p$ can be replaced by $2^{p-1}$;
   
   (b) if $0 \leq p \leq 1$, the factor $2^p$ can be replaced by $1$.

4. If $p > 1$, we have
   
   $$\left|\frac{1}{n} \sum_{i=1}^{n} X_i\right|^p \leq \frac{1}{n} \sum_{i=1}^{n} |X_i|^p,$$
   
   so that
   
   $$E\left[\left|\frac{1}{n} \sum_{i=1}^{n} X_i\right|^p\right] \leq \frac{1}{n} \sum_{i=1}^{n} E[|X_i|^p];$$
   
   we have also
   
   $$E\left[\left|\frac{1}{n} \sum_{i=1}^{n} X_i\right|^p\right] \leq \left\{\frac{1}{n} \sum_{i=1}^{n} \left(E[|X_i|^p]\right)^{1/p}\right\}^p.$$