These questions are only meant as a study aid and to help you test your knowledge. Being able to solve them does not guarantee that you are well-prepared for the exam.

1. For each of the following joint densities, indicate whether $Y_1$ and $Y_2$ are independent (Yes or No). No explanation is required.

   (a) $f(y_1, y_2) = \begin{cases} 2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, 0 \leq y_1 + y_2 \leq 1, \\ 0, & \text{elsewhere}. \end{cases}$

   (b) $f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 \geq 0, y_2 \geq 0, \\ 0, & \text{elsewhere}. \end{cases}$

   (c) $f(y_1, y_2) = \begin{cases} y_1 + y_2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere}. \end{cases}$

   (d) $f(y_1, y_2) = \begin{cases} e^{-y_1}, & y_1 \geq 0, 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere}. \end{cases}$

   (e) $f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty, \\ 0, & \text{elsewhere}. \end{cases}$

2. Let $Y_1$ and $Y_2$ denote the proportion of time, out of one workday, that employees I and II, respectively, actually spend performing their assigned tasks. The joint relative frequency behavior of $Y_1$ and $Y_2$ is modeled by the density function

   $f(y_1, y_2) = \begin{cases} y_1 + y_2, & 0 \leq y_1 \leq 1; 0 \leq y_2 \leq 1 \\ 0, & \text{elsewhere}. \end{cases}$

   (a) Find $P(Y_1 \geq \frac{1}{2}, Y_2 \geq \frac{1}{2})$.

   (b) Find the marginal density function for $Y_2$.

   (c) Find $P(Y_2 \geq \frac{1}{2})$.

   (d) Find $P(Y_1 \geq \frac{1}{2} | Y_2 \geq \frac{1}{2})$.

   (e) Find the conditional density of $Y_1$ given $Y_2$.

   (f) Find $P(Y_1 \geq \frac{1}{2} | Y_2 = \frac{1}{4})$.

3. Suppose that the random variables $Y_1$ and $Y_2$ have joint density given by

   $f(y_1, y_2) = \begin{cases} \frac{1}{2} + 2y_1y_2, & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere}. \end{cases}$

   Note that $f(y_1, y_2)$ is symmetric in $y_1$ and $y_2$, so that $Y_1$ and $Y_2$ have the same marginal distributions, and hence the same mean and variance.

   (a) Find $E(Y_1)$.

   (b) Find $V(Y_1)$.

   (c) Find Cov($Y_1, Y_2$).

   (d) Find $E(2Y_1 + 2Y_2 + 5)$.

   (e) Find $V(2Y_1 + 2Y_2 + 5)$.

   (f) Find Cov($2Y_1 - Y_2, Y_1 + Y_2$).
4. Suppose that \( V(Y_1) = V(Y_2) = \sigma^2 \) and \( \text{Cov}(Y_1, Y_2) = \gamma \). Find
   (a) \( V(Y_1 + Y_2) \)
   (b) \( V(Y_1 - Y_2) \)
   (c) \( \text{Cov}(Y_1 + Y_2, Y_1 - Y_2) \)

5. Suppose that \( Z \sim N(0, 1) \) and \( X \sim \chi^2_\nu \) are independent, and let
   \[ T = \frac{Z}{\sqrt{X/\nu}}. \]
   Find \( E(T) \) and \( V(T) \). What must you assume about \( \nu \)?

6. Suppose that \( X_1 \sim \chi^2_{\nu_1} \) and \( X_2 \sim \chi^2_{\nu_2} \) are independent, and let
   \[ F = \frac{X_1/\nu_1}{X_2/\nu_2}. \]
   Find \( E(F) \) and \( V(F) \). What must you assume about \( \nu_1 \) or \( \nu_2 \)?

7. Let the random variable \( Y \) have density
   \[ f(y) = \begin{cases} \frac{1}{2}(1 + y), & -1 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases} \]
   Find the density of \( W = Y^2 \).

8. Let \( Y_1 \) and \( Y_2 \) have joint density
   \[ f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{elsewhere,} \end{cases} \]
   and let \( W = Y_1 - Y_2 \). Find the density of \( W \).

9. The Romulans have trapped the starship Enterprise in a spherical energy bubble. Spock has determined that at any given time, the radius of the bubble, \( Y \), is a random variable with probability density function
   \[ f_Y(y) = \begin{cases} \frac{72y^2}{(9 + 16\pi^2y^6)}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases} \]
   In order to plan an escape, Captain Kirk desperately needs to know the probability density function of the volume, \( V \), of the bubble. Knowing that the volume of a sphere of radius \( Y \) is given by the formula
   \[ V = \frac{4}{3}\pi Y^3, \]
   Spock uses the method of transformations to find the probability density function of \( V \). Since Captain Kirk's algebra is not very good, Spock also simplifies his result. What is Spock's answer?

10. Suppose that \( Y_1, \ldots, Y_5 \) are independent, exponentially distributed random variables, each with mean \( \beta = 10 \). Use the method of moment generating functions, or results derived in class, to find the density of
    \[ \overline{Y} = \frac{1}{5}(Y_1 + Y_2 + Y_3 + Y_4 + Y_5). \]
11. Let \( Y_1 \) and \( Y_2 \) be independent Poisson random variables with means \( \lambda_1 \) and \( \lambda_2 \), respectively.

   (a) Use the method of moment generating functions to show that the distribution of \( W = Y_1 + Y_2 \) is Poisson with mean \( \lambda_1 + \lambda_2 \).

   (b) Use the result of part (a) to show that the conditional probability function of \( Y_1 \) given \( W = w \) is binomial with \( n = w \) and \( p = \lambda_1 / (\lambda_1 + \lambda_2) \). *Hint:* \( \Pr(Y_1 = y_1, W = w) = \Pr(Y_1 = y_1, Y_2 = w - y_1) \).

12. Let \( Y_1, \ldots, Y_{10} \) be a random sample from the distribution with density function

   \[ f(y) = \begin{cases} 
   \frac{2y}{\theta^2}, & 0 \leq y \leq \theta, \\
   0, & \text{elsewhere},
   \end{cases} \]

   where \( \theta > 0 \).

   (a) Find the density of \( X = \max(Y_1, \ldots, Y_{10}) \).

   (b) Find \( E(X) \) and \( \text{Var}(X) \).

13. Consider the probability density function

   \[ f(y) = \begin{cases} 
   0, & y < 0, \\
   (1 + y)^{-2}, & y \geq 0.
   \end{cases} \]

   Suppose that \( U \sim U(0, 1) \). Find a transformation \( h(U) \) so that \( Y = h(U) \) has density \( f \).

14. Short answer. You are not required to show any work.

   (a) Suppose that \( Y_1, Y_2, \) and \( Y_3 \) are independent exponential random variables, each having mean 10, and let \( X = Y_1 + Y_2 + Y_3 \). Name the distribution of \( X \) and give the values of any parameters.

   (b) Refer to part (b). For what value of \( c \) does \( W = cX \) have a chi-square distribution? With how many degrees of freedom?

15. Short answer. You are not required to show any work.

   Suppose that \( Y_1 \) and \( Y_2 \) are independent normal random variables each with mean \( \mu \) and variance \( \sigma^2 \). Name the distribution of each of the following random variables and give the values of any parameters.

   (a) \( X = Y_1 + Y_2 \)

   (b) \( W = \left( \frac{Y_1 - \mu}{\sigma} \right)^2 + \left( \frac{Y_2 - \mu}{\sigma} \right)^2 \)

   (c) \( V = \left( \frac{Y_1 + Y_2 - 2\mu}{\sqrt{2\sigma^2}} \right)^2 \) *Hint:* Refer to part (a).

16. Let \( Y \sim \text{Beta}(1, \theta) \), so that \( Y \) has probability density function

   \[ f(y) = \theta(1 - y)^{\theta - 1}, \quad 0 < y < 1. \]

   (a) Show that \( X = -\ln(1 - Y) \sim \text{Exp}(\beta = 1/\theta) \).

   (b) Now suppose that \( Y_1, \ldots, Y_n \) are independent random variables, each having a \( \text{Beta}(1, \theta) \) distribution. Name the distribution of \( U = \sum_{i=1}^{n}[-\ln(1 - Y_i)] \) and give the values of any parameters.

   (c) In the situation of part (b), find \( E(1/U) \).