LEcTUrE - 20

Agenda:

Examples: Maximum likelihood estimate (MLE).

Example 1: Suppose \( y_1, y_2, \ldots, y_n \) is a random sample from a population which is Exponential (\( \theta \)). Provide the Maximum likelihood estimate of \( \theta \).

Step 1: Write down the likelihood at \( y_1, y_2, \ldots, y_n \)

\[
L(y_1, y_2, \ldots, y_n | \theta) = \prod_{i=1}^{n} f_{Y_i}(y_i) \quad (\text{By independence})
\]

\[
= \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{y_i}{\theta}} = \left( \frac{1}{\theta} \right)^n e^{-\frac{1}{\theta} \sum_{i=1}^{n} y_i}
\]

Step 2: Find the value of \( \theta \) which maximizes the likelihood.
We want to maximize the function

\[ L(\theta) \text{ (shortened for } L(y_1, y_2, \ldots, y_n | \theta)) \]

**Simplifying Rule:** If \( L(\theta) \) contains an exponential term, it is often useful to try and maximize \( \log L(\theta) \) instead of \( L(\theta) \). Note that if \( \hat{\theta} \) maximizes \( \log L(\theta) \), it also maximizes \( L(\theta) \).

So let us maximize

\[
\log L(\theta) = \log \left( \frac{1}{\theta^n} e^{-\frac{y_i^2}{2\theta}} \right) = -n \log \theta - \frac{y_i^2}{2\theta}
\]

Using calculus, let us solve \( \frac{d}{d\theta} \log L(\theta) = 0 \).

\[
\frac{d}{d\theta} \log L(\theta) = 0
\]

\[
-\frac{n}{\theta} + \frac{y_i^2}{2\theta} = 0
\]

\[
\theta = \left( \frac{1}{\sum_{i=1}^{n} y_i^2} \right)^n
\]
It is easy to check that
\[
\frac{\partial^2}{\partial \theta^2} \log L(\theta) \bigg|_{\theta = \frac{\sum y_i}{n}} = -\frac{3}{(\frac{\sum y_i}{n})^2} < 0.
\]

Hence \( \log L(\theta) \) is maximized at \( \theta = \frac{\sum y_i}{n} \). The MLE is \[ \hat{\theta} = \frac{\sum y_i}{n}. \]

Example 2: Suppose \( y_1, y_2, \ldots, y_n \) is a random sample from a population which is Uniform \((0, \theta)\). Find the maximum likelihood estimate of \( \theta \).

Steps: Write down the likelihood at an arbitrary possible configuration \( y_1, y_2, \ldots, y_n \).

\[
L(y_1, y_2, \ldots, y_n) = \frac{\prod_{i=1}^{n} f_{Y_i}(y_i)}{f_{Y_1}(y_1) \cdots f_{Y_n}(y_n)}
\]

\[
= \frac{\prod_{i=1}^{n}}{f_{Y_i}(y_i)}
\]

\[
= \frac{1}{\theta^n} \frac{1}{\theta} \frac{1}{\{0 \leq y_i \leq \theta\}}
\]

\[
= \frac{1}{\theta^n} \prod_{i=1}^{n} \frac{1}{\theta} \{0 \leq y_i \leq \theta\}
\]
\[
\frac{1}{\theta^n} \prod_{i=1}^{n} \left\{ 0 \leq y_i \leq \theta \text{ for every } i=1,\ldots,n \right\}
\]

\[
\frac{1}{\theta^n} \prod_{i=1}^{n} \left\{ \max(y_1, y_2, \ldots, y_n) \leq \theta, \min(y_1, y_2, \ldots, y_n) \geq 0 \right\}
\]

\[
\frac{1}{\theta^n} \prod_{i=1}^{n} \left\{ \max(y_1, y_2, \ldots, y_n) \leq \theta \right\} \frac{1}{\theta^n} \prod_{i=1}^{n} \left\{ \min(y_1, y_2, \ldots, y_n) \geq 0 \right\}
\]

**Step 2:** Find the value of \(\theta\) which maximizes the likelihood.

\[
L(\theta) = \frac{1}{\theta^n} \prod_{i=1}^{n} \left\{ \max(y_1, y_2, \ldots, y_n) \leq \theta \right\} \frac{1}{\theta^n} \prod_{i=1}^{n} \left\{ \min(y_1, y_2, \ldots, y_n) \geq 0 \right\}
\]

Note that \(L(\theta) = 0\) if \(0 < \max(y_1, y_2, \ldots, y_n)\).

Hence the value of \(\theta\) maximizing \(L(\theta)\) must lie in the interval \([\max(y_1, y_2, \ldots, y_n), \infty)\), since \(L(\theta)\) is a non-negative function. On this interval, \(L(\theta) = \frac{1}{\theta^n} \prod_{i=1}^{n} \left\{ \min(y_1, y_2, \ldots, y_n) \geq 0 \right\}\).

The function \(\frac{1}{\theta^n}\) is a decreasing function on \([\max(y_1, y_2, \ldots, y_n), \infty)\). Hence, \(L(\theta)\) is maximized at \(\theta = \max(y_1, y_2, \ldots, y_n)\).

The MLE is \(\hat{\theta} = \max(y_1, y_2, \ldots, y_n)\).