LECTURE 6

Agenda:

1. Inclusion - Exclusion principle
2. Theorem of total probability
3. Bayes rule

INCLUSION-EXCLUSION PRINCIPLE

The inclusion - exclusion principle provides an identity for computing the probability of union of a set of events in terms of intersections of various orders of these events.

2 events: If A, B are two events,

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B).$$

3 events: If A, B, C are three events, then

$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(B \cap C)$$
$$- \Pr(A \cap C) - \Pr(A \cap B \cap C).$$
**k events:** If $A_1, A_2, \ldots, A_k$ are $k$ events,

\[
P(A_1 \cup A_2 \cup \ldots \cup A_k) = \sum_{i=1}^{k} P(A_i) - \sum_{\text{all unordered}} P(A_{i_1} \cap A_{i_2}) + \sum_{\text{all unordered}} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \cdots + (-1)^{k-1} P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k})
\]

**Example:** An absent-minded secretary prepared five letters and envelopes to send to five different people. Then she randomly placed letters in the envelopes. A match occurs if the letter and its envelope are addressed to the same person. What is the probability that at least one of the five letters and envelopes match?

Let, $A =$ Event that at least one match occurs.

$A_i =$ Event that letter $i$ is placed in envelope $i$ for $i = 1, 2, 3, 4, 5.$

Then, $A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$

Note that all arrangements of the 5 letters are equally likely.
Note that the total number of arrangements possible is $5!$.

The number of arrangements in the event $A_i$ is $4!$ for $i = 1, 2, 3, 4, 5$ (Why?)

The number of arrangements in the event $A_{i_1} \cap A_{i_2}$ is $3!$ for all unordered pairs $(i_1, i_2)$ (Why?)

The number of arrangements in the event $A_{i_1} \cap A_{i_2} \cap A_{i_3}$ is $2!$ for all unordered triplets $(i_1, i_2, i_3)$ (Why?)

The number of arrangements in the event $A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4}$ is $1!$ for all unordered quadruplets $(i_1, i_2, i_3, i_4)$ (Why?)

The number of arrangements in the event $A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$ is $1$.

Hence, by the inclusion-exclusion principle,

$$P(A) = \frac{5!}{5!} - \sum_{\text{all unordered pairs } (i_1, i_2)} \frac{3!}{5!} + \sum_{\text{all unordered triplets } (i_1, i_2, i_3)} \frac{2!}{5!}$$

$$- \sum_{\text{all unordered quadruplets } (i_1, i_2, i_3, i_4)} \frac{1!}{5!} + \frac{1}{5!}$$

$$= 5 \cdot 4! - \binom{5}{2} \frac{3!}{5!} + \binom{5}{3} \frac{2!}{5!} - \binom{5}{4} \frac{1!}{5!} + \frac{1}{5!}$$
Hence, \[ P(A) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} = 0.6417 \]

**THEOREM OF TOTAL PROBABILITY**

If \( B_1, B_2, \ldots, B_k \) is a collection of mutually exclusive and exhaustive events, then for any event \( A \),

\[ P(A) = \sum_{i=1}^{k} P(B_i) P(A \mid B_i) \]

**Example:** A company buys microchips from three suppliers. Supplier I microchips have 20% chance of being defective, Supplier II microchips have 5% chance and Supplier III microchips have 2% chance of being defective. Suppose 25%, 35% and 40% of the current supply came from Suppliers I, II, III respectively. If a microchip is selected randomly from this supply, what is the probability that it is defective?

\[ B_1 = \text{Event that microchip comes from Supplier I} \]
\[ B_2 = \text{Event that microchip comes from Supplier II} \]
\[ B_3 = \text{Event that microchip comes from Supplier III} \]
\[ A = \text{Event that microchip is defective} \]
It is given that

\[ P(A \mid B_1) = 0.1, \quad P(A \mid B_2) = 0.05, \quad P(A \mid B_3) = 0.02 \]

Note that \( B_1, B_2, B_3 \) are mutually exclusive and exhaustive, since \( B_1 \cup B_2 \cup B_3 = S \), and \( B_i \cap B_j = \phi \) for \( i \neq j \). Also,

\[ P(B_1) = 0.2, \quad P(B_2) = 0.35, \quad P(B_3) = 0.45. \]

Hence, by the Theorem of total probability,

\[ P(A) = 0.1 \times 0.2 + 0.05 \times 0.35 + 0.02 \times 0.45 = 0.046 \]

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**Bayes' Rule**

If the events \( B_1, B_2, \ldots, B_k \) are mutually exclusive and exhaustive, then for any event \( A \),

\[
P(B_i \mid A) = \frac{P(A \mid B_i) P(B_i)}{\sum_{j=1}^{k} P(A \mid B_j) P(B_j)}.
\]
Example: In the previous example, if a randomly selected microchip is defective, what is the probability that it came from supplier II?

By Bayes rule,

\[
P(B_2 | A) = \frac{P(A | B_2) P(B_2)}{\sum_{i=1}^{3} P(A | B_i) P(B_i)}
\]

\[
= \frac{0.05 \times 0.35}{0.0465}
\]

\[
= 0.376.
\]