LEcTURe 5

Agenda:

1. Conditional Probability
2. Independence

CONDITIONAL PROBABILITY

Many times, some partial information about the outcome of a random experiment is available and we want to revise our estimates of the chance of an event accordingly.

Definition: Let A and B be two events in a random experiment with sample space S. Then, the probability of the event A given that the event B has occurred is defined as

\[ P(A \mid B) \triangleq \frac{P(AB)}{P(B)} \]

provided that \( P(B) > 0 \).

The symbol \( P(A \mid B) \) is read "the probability of A given B".
Example: Toss a fair die. Let $A$ denote the event that the outcome is 2, 4 or 6. If somebody asks you to play the following game:

"If event $A$ occurs, you pay me $10, otherwise I will pay you $10."

Will you play the game?

Ans: $S = \{1, 2, 3, 4, 5, 6\}$

All outcomes are equally likely, as it is a fair die with no preference for any outcome.

$A = \{2, 4, 6\}$.

Hence, $P(A) = \frac{3}{6} = \frac{1}{2}$.

It seems like a fair game and you would bet.

Suppose the die is cast in a secret chamber and you have a helpful source who tells you that the result was 4, 5 or 6. You have a chance to withdraw or remain in the game. What would you do?
Ans: \( B = \{4, 5, 6\} \). We are told that \( B \) has occurred. Let us evaluate \( P(A|B) \).

\[
P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{4, 6\})}{P(\{4, 5, 6\})} = \frac{2}{3}.
\]

Hence, the chance that you will have to play the other person $10 is \( \frac{2}{3} \) given this additional information. You would withdraw and not play the game.

Conditional probabilities satisfy the 3 axioms of probability.

**Result:** Let \( B \) be an event with \( P(B) > 0 \). Then,

1. \( 0 \leq P(A|B) \leq 1 \) for every event \( A \).
2. \( P(\emptyset|B) = 0 \).
3. If \( A_1, A_2, A_3, \text{---} \) are mutually exclusive events, then

\[
P(\bigcup_{i=1}^{\infty} A_i | B) = \sum_{i=1}^{\infty} P(A_i | B).
\]
Proof: 1) Note that $0 \leq P(AB) \leq P(B)$.

Dividing everything by $P(B)$, we get that

$$0 \leq \frac{P(AB)}{P(B)} \leq 1$$

$\implies 0 \leq P(AB \mid B) \leq 1$.

2) $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$.

3) $P(\bigcup_{i=1}^{\infty} A_i \mid B) = \frac{P(\bigcup_{i=1}^{\infty} (A_i \cap B))}{P(B)}$

$= \frac{P(\bigcup_{i=1}^{\infty} (A_i \cap B))}{P(B)}$

(\because \text{ By distributive laws})

$= \sum_{i=1}^{\infty} \frac{P(A_i \cap B)}{P(B)}$

(\because A_1 \cap B, A_2 \cap B, A_3 \cap B, \ldots \text{ are mutually exclusive})

$= \sum_{i=1}^{\infty} P(A_i \mid B)$. 
INDEPENDENCE

If the extra information provided by knowing that an event B has occurred does not change the probability of A, i.e., if $P(A|B) = P(A)$, then the events A and B are said to be independent.

**Definition:** Two events A and B are said to be independent if

$$P(A \cap B) = P(A)P(B).$$

- $P(A \cap B) = P(A)P(B)$ is equivalent to stating that $P(A|B) = P(A)$ or $P(B|A) = P(B)$, if the conditional probabilities $P(A|B)$ or $P(B|A)$ exist.

- **Multiplicative rule:** If $A_1, A_2, \ldots, A_n$ are n events, then

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2|A_1)\cdots P(A_n|A_1A_2\cdots A_{n-1}) \times P(A_n|A_1A_2\cdots A_{n-1}).$$

**Example:** Draw a card from a shuffled 52-card deck with no preference to any card. Let A denote the event that a king is drawn, and let B denote the event
that a diamond ♦ is drawn.

\[ P(A) = \frac{4}{52}, \quad P(B) = \frac{4 \cdot 3}{52}, \quad P(A \cap B) = \frac{1}{52}. \]

Hence, \( P(A \cap B) = P(A)P(B) \). This gives independence of \( A \) and \( B \).