Agenda:

1. Beta distribution
2. Moment generating function

BETA DISTRIBUTION

Every continuous distribution that we have encountered except the uniform, takes values over an infinite interval \((0, \infty)\) or \(\mathbb{R}\). The beta distribution is an alternative model for random variables which are constrained to lie in the interval \((0, 1)\).

A random variable \(X\) is said to have a Beta\((\alpha, \beta)\) distribution (where \(\alpha\) and \(\beta\) are fixed constants) if

\[
X = \text{Range}(X) = (0, 1)
\]

\[
f_X(x) = \begin{cases} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x \notin [0, 1]. \end{cases}
\]
The first thing that we need to check is
\[ \sum_{x=\infty}^{\infty} f_x(x) dx = 1. \]

To prove this, we use the following identity.

**RESULT:** If \( \alpha, \beta \) are positive constants, then
\[
\int_{0}^{1} x^{\alpha-1} (1-x)^{\beta-1} dx = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)}
\]

The proof of this identity is omitted. Hence,
\[
\sum_{x=\infty}^{\infty} f_x(x) dx = \int_{0}^{1} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \Gamma(\beta)} \cdot x^{\alpha-1} (1-x)^{\beta-1} dx
\]

\[= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \Gamma(\beta)} \cdot \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)} = 1. \]

The distribution function \( F_X \) is not available in closed form, but can be evaluated using software packages. The Beta density can take a variety of shapes which points to the flexibility of this collection of random variables.
**RESULT:** The Uniform \([0,1]\) random variable is a special case of the Beta random variable with \(\alpha = 1, \beta = 1\).

The graph of the Beta density for four possible situations is illustrated.
Let us calculate $E(X)$.

$$E(X) = \int_{-\infty}^{\infty} x f_{X}(x) \, dx$$

$$= \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_{0}^{\infty} x \alpha \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} \, dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \Gamma(\beta)} \int_{0}^{\infty} x \alpha (1-x)^{\beta-1} \, dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \Gamma(\beta)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \Gamma(\beta)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+2)}$$

$$= \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+2)}$$

($\beta$: by the properties of the Gamma function).

Similarly,

$$V(X) = \frac{\beta}{\alpha+\beta}$$

**Example:** A gasoline wholesale distributor uses bulk storage tanks to hold a fixed supply. The tanks are
filled every Monday. Of interest to the wholesaler is the proportion of the supply sold during the week. Every many weeks, this proportion has been observed to match fairly well a beta distribution with \( \alpha = 4 \) and \( \beta = 2 \).

(b) Find the expected value of this proportion.

\[
E(X) = \frac{\alpha}{\alpha + \beta} = \frac{4}{4 + 2} = \frac{2}{3}.
\]

(b) Find the probability that the wholesaler will sell at least 90% of the stock in a given week.

\[
P(X \geq 0.9) = \int_{0.9}^{\infty} f_X(x) \, dx
\]

\[
= \int_{0.9}^{\infty} \frac{\beta^\alpha}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \, dx
\]

\[
= \left[ \frac{1}{2} \right] \int_{0.9}^{1} x^3(1-x) \, dx
\]

(C : \( \Gamma(x) = (x-1)! \) if \( x \) is a nonnegative integer)

\[
= 20 \int_{0.9}^{1} (x^4 - x^3) \, dx
\]

\[
= 20 \left( 0.049 \right)
\]
Hence there is an 81% chance that the wholesaler will sell at least 90% of the stock in a given week.

**MOMENT GENERATING FUNCTION**

For a random variable $X$, (discrete or continuous), the moment generating function is a special function associated to it. The moment generating function is a powerful theoretical tool. The name "moment generating function" comes from the fact that it can be used to obtain the "moments" of the random variable. We clarify this shortly.

**Definition:** If $X$ is a discrete or continuous random variable, then the moment generating function of $X$ is denoted by $M_X : \mathbb{R} \to (0, \infty)$, and is defined by

$$M_X(t) = E[e^{tX}] = \sum_{x \in \mathbb{X}} e^{tx} p_X(x)$$

if $X$ is discrete,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) \, dx$$

if $X$ is continuous.

**IMPORTANT:** The moment generating function is defined for only those $t$, where the sums or integrals
considered in the definition exist.

For a random variable $X$, the values $E(X), E(X^2), E(X^3)$ are known as the moments of the random variable. They provide useful information about the random variable.

**RESULT:** \[ \frac{d}{dt} M_X(t) \bigg|_{t=0} = E(X). \]

\[ \frac{d^2}{dt^2} M_X(t) \bigg|_{t=0} = E(X^2) \]

In fact, for any positive integer $k$,

\[ \frac{d^k}{dt^k} M_X(t) \bigg|_{t=0} = E(X^k). \]

Hence the moment generating function can be used to compute the moments of the random variable (provided that the moment generating function is well defined in an interval around zero).