LECTURE 11

Agenda:

1. Tchebysheff's theorem (from Lecture 10)
2. Bernoulli random variables

Tchebysheff's theorem is typically quite conservative, but that's the price to pay for a general result.

Example: Let $X = \#$ of heads in 3 tosses of a fair coin (independently)

Then, $P(X = 0) = \frac{1}{8}$, $P(X = 1) = \frac{3}{8}$, $P(X = 2) = \frac{3}{8}$, $P(X = 3) = \frac{1}{8}$.

$E(X) = \frac{3}{2}$, $V(X) = \frac{3}{4}$.

Applying Tchebysheff's theorem with $k = 2$, we get that,

$P\left( \left| X - \frac{3}{2} \right| < \sqrt{3} \right) \geq 1 - \frac{1}{2^2}$

i.e. $P(-0.232 < X < 3.232) \geq 0.75$

But this is very conservative.
WARNING: For a discrete random variable X,

\[ E(X) = \sum_{x \in X} x \cdot p_X(x) = \sum_{x \in X} xP(X=x) \]

is well-defined only if \( \sum_{x \in X} |x| \cdot p_X(x) < \infty \).

If \( X \) is a finite space, this condition will always hold. However, when \( X \) is not a finite space, one should always make sure that the expectations are finite.

**Example:** Suppose \( X \) is a random variable with range \( X = \{1, 2, 3, \ldots, \infty\} \), and the probability mass function of \( X \) is given by

\[ P(X=x) = \frac{6}{\pi^2 x^2} \quad \text{for every } x \in X. \]

(Note that \( \sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6} \), hence the above assignment of probabilities is valid.)

Then,

\[ \sum_{x \in X} |x| \cdot p_X(x) = \sum_{x=1}^{\infty} \frac{1}{x} = \infty. \]

Hence \( E(X) \) is not well-defined in this case.
BERNOULLI RANDOM VARIABLES

An experiment with two possible outcomes is called a "Bernoulli experiment," or a "Bernoulli trial." Suppose one outcome of a Bernoulli trial is identified as success and the other outcome is identified as failure. Define the random variable $X$ such that

$$X = \begin{cases} 
1 & \text{if the outcome of the trial is a success,} \\
0 & \text{otherwise.}
\end{cases}$$

Let $p$ denote the probability of success in the experiment, i.e.,

$$P(X = 0) = 1 - p, \quad P(X = 1) = p.$$  

Such a random variable is said to be a "Bernoulli random variable." The only parameter needed to describe the probability distribution of this random variable is $p$.

$$E(X) = 0 \cdot (1-p) + 1 \cdot p = p.$$  

$$V(X) = E(X^2) - (E(X))^2$$
$$= 0 \cdot (1-p) + 1 \cdot p - p^2$$
$$= p(1-p).$$