Lecture 24

Agenda

1. Examples from Normal Distribution
2. Beta distribution
3. Moment Generating Function

Example

Mainly two kind of examples are done for normal distribution.

Example 1

Suppose that men’s neck sizes are approximately normally distributed with a mean of 16.2 inches and variance of 0.81 square inch. Find the probability that the neck size of a randomly chosen man lies between 13.5 and 18.9 inches.

Let $X$ = Men’s neck size in inches. Then $X \sim N(\mu = 16.2, \sigma^2 = 0.81)$.

\[
P(13.5 \leq X \leq 18.9) = P(X \leq 18.9) - P(X \leq 13.5)
= \Phi \left( \frac{18.9 - 16.2}{0.9} \right) - \Phi \left( \frac{13.5 - 16.2}{0.9} \right)
= \Phi (3) - \Phi (-3) = 0.997
\]

Example 2

Suppose scores in an exam follow normal distribution with mean 80 and standard deviation 5. What’s the minimum score that you should get to be in the top 10%?

Let $X$ be the score of a randomly chosen student and suppose you have to score minimum $x$ to be in the top 10%.

\[
X \sim N(\mu = 80, \sigma^2 = 5^2)
\]

Then $P(X \leq x) = 0.9$.

We also know $P(X \leq x) = \Phi \left( \frac{x-\mu}{\sigma} \right) = \Phi \left( \frac{x-80}{5} \right)$. Hence $\Phi \left( \frac{x-80}{5} \right) = 0.9$. Now from computers I can find out that $\Phi^{-1}(0.9) = 1.2815$. Hence $\frac{x-80}{5} = 1.2815$, i.e. $x = 86.40776$.  

Beta Distribution

Every continuous distribution which we have encountered except the uniform distribution, takes values over an infinite interval like \((0, \infty)\) or \(\mathbb{R}\). The beta distribution is an alternative model for random variables which can be constrained in the interval \((0, 1)\). In general if \(X\) can be constrained in the interval \((a, b)\) then \(Y = \frac{X-a}{b-a}\) can be constrained in \((0, 1)\) and thus by studying properties of \(Y\) we can study properties of \(X\).

**Definition 1.** A random variable \(X\) is said to follow the **Beta distribution** with parameters \((\alpha, \beta)\) for some \(\alpha > 0\) and \(\beta > 0\) if,

\[
\text{Range}(X) = (0, 1)
\]

and for \(x \in (0, 1)\)

\[
f_X(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}
\]

We write this as \(X \sim \text{Beta}(\alpha, \beta)\)

The first thing that we need to check is that \(\int_0^1 f_X(x)\,dx = 1\), i.e. we need to check the following identity for \(\alpha, \beta > 0\).

\[
\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad \ldots (\ast)
\]

but we are omitting this proof for now. Instead let’s do the mean and variance,
Mean and Variance

\[ E(X) = \int_0^1 x f_X(x) dx \]
\[ = \int_0^1 x \times \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx \]
\[ = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha}(1-x)^{\beta-1} dx \]
\[ = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \frac{\Gamma(\alpha + 1)\Gamma(\beta)}{\Gamma((\alpha + 1) + \beta)} \quad \text{[from (*)]} \]
\[ = \frac{\Gamma(\alpha + \beta)}{\Gamma((\alpha + 1) + \beta)} \times \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)} \]
\[ = \frac{\Gamma(\alpha + \beta)}{(\alpha + \beta)\Gamma(\alpha + \beta)} \times \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)} \]
\[ = \frac{\alpha}{\alpha + \beta} \]

Similarly,

\[ V(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}. \]

How does the density look like?

We have plotted the density for four combinations of \( \alpha \) and \( \beta \). More will be discussed in lecture. Please note that \( \alpha = \beta = 1 \) means the uniform density.
Moment generating function

For a random variable $X$ (discrete or continuous), the moment generating function is a special function associated with it. The moment generating function although intuitively looks strange at first is a powerful theoretical tool. The name “moment generating function” comes from the fact that it can be used to generate the “moments” of $X$. We clarify this shortly.
For any $t \in \mathbb{R}$,

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \quad \text{[If } X \text{ is continuous]}$$

$$= \sum_{x \in \text{Range}(X)} e^{tx} P(X = x) \quad \text{[If } X \text{ is discrete]}$$

Now if $t \neq 0$, there is no guarantee that the above sum or integral will be finite.

**Definition 2.** Let $X$ be a random variable and $A = \{t : E(e^{tX}) < \infty\}$. Then we define the moment generating function of $X$, as the function $M_X : A \to (0, \infty)$ where for $t \in A$,

$$M_X(t) = E(e^{tX})$$

Now for any random variable $X$,

$$E(X), E(X^2), E(X^3), \ldots$$

are known as the moments of the random variable. They provide useful information about the random variable. The following result tells us why this function is called the moment generating function.

**Lemma 1.** If for a random variable $X$, $M_X$ can be defined for all values in any interval $(-\epsilon, \epsilon)$ around 0 then for $k \geq 1$,

$$\frac{d^k}{dt^k} M_X(t) \bigg|_{t=0} = E(X^k)$$

Thus the moment generating function can be used to generate moments.

**Homework::**

1. If $X \sim \text{Beta}(\alpha, \beta)$, then $Y = 1 - X \sim \text{Beta}(\beta, \alpha)$.

2. Prove the formula for variance of beta distribution.

3. 4.123 a, 4.124, 4.125