Lecture 12

Agenda
1. Bernoulli Random Variable
2. Binomial Distribution

Bernoulli Random Variable

**Definition 1.** If the random variable $X$ has the following distribution

$$P(X = 1) = p, \quad P(X = 0) = 1 - p$$

for some $0 < p < 1$, then $X$ is called a Bernoulli random variable and we write

$$X \sim Ber(p)$$

**Example**

Suppose an experiment has only two outcomes; let’s call one of the outcomes as success and the other one as failure. *(The words success and failure may not have any real meaning.)* Then define

$$X = 1 \text{ if success occurs}$$
$$= 0 \text{ if failure occurs}$$

We see that $X$ is a bernoulli random variable and $X \sim Ber(p)$, where $p$ is the probability of success.

If an experiment has only two possible outcomes like this one, we call it a Bernoulli trial.
Examples of Bernoulli trial include —— tossing a coin once, Gators winning or losing the next game, whether it’s gonna rain tomorrow etc.

**Pmf**

$$Range(X) = \{0, 1\}$$

$$p_X(1) = p, p_X(0) = 1 - p$$

Observe that we can also write the pmf as

$$p_X(x) = p^x(1 - p)^{1-x}$$

for $x \in \{0, 1\}$. 
Mean and Variance

Let $X \sim Ber(p)$.

$$E(X) = 1 \times p + 0 \times (1 - p) = p$$

$$E(X^2) = 1^2 \times p + 0^2 \times (1 - p) = p$$

$$V(X) = E(X^2) - [E(X)]^2 = p - p^2 = p(1 - p)$$

Binomial Distribution

Let’s begin with the following example. I have a coin where $P(Heads) = p$ and $P(Tails) = (1 - p)$.

**Toss it once:**

Let $X$ denote the number of heads. Then $X \sim Ber(p)$, because you can either get heads($X = 1$) or tails($X = 0$) with probabilities $p$ and $(1 - p)$.

**Toss it n times:**

Let $X$ denote the number of heads. Then $Range(X) = \{0, 1, 2, \ldots, n\}$.

For $k \in \{0, 1, 2, \ldots, n\}$,

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{(n-k)}$$

**WHY ?**

$X = k$ corresponds to $\binom{n}{k}$ many outcomes; because, out of the $n$ tosses choose the $k$ tosses where you will have heads in $\binom{n}{k}$ many ways, and put the tails in the remaining $(n - k)$ positions. For each of these outcomes we have $k$ heads and $(n - k)$ tails and hence the probability for each of those outcomes $= p^k (1 - p)^{(n-k)}$

**Definition 2.** For $0 < p < 1$ and a number $n \geq 1$, a random variable $X$ is said to follow the **Binomial distribution** with parameters $n$ and $p$; [written as $X \sim Bin(n, p)$] if

$$Range(X) = \{0, 1, 2, \ldots, n\}$$

and for $0 \leq k \leq n$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{(n-k)}$$
Natural situation for binomial distribution

Suppose there is a Bernoulli experiment with probability of success $p$ and probability of failure $(1-p)$. You are repeating the experiment independently $n$ times. Then $X$ the number of successes, follow $Bin(n, p)$. In fact suppose for $i = 1, 2, \ldots, n$, let

$$X_i = \begin{cases} 1 & \text{if } i\text{-th experiment results in success} \\ 0 & \text{if } i\text{-th experiment results in failure} \end{cases}$$

then we observe that

$$X = X_1 + X_2 + \ldots + X_n$$

and

$$X \sim Bin(n, p)$$

Pmf

Let $X \sim Bin(n,p)$. For $x \in \{0,1,2,\ldots,n\}$

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Observe one thing that we got free out of all this

**Lemma 1.** If $0 < p < 1$ and $n \geq 1$,

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$$

Mean and Variance

If $X \sim Bin(n, p)$, then

$$E(X) = np$$
$$V(X) = np(1-p)$$

**Proof.**

$$E(X) = E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n) = np$$

since each $X_i \sim Ber(p)$
Now there is a result that says that if $Y_1, Y_2, \ldots, Y_k$ arised out of $k$ independent experiments then

$$V(Y_1 + Y_2 + \ldots + Y_k) = V(Y_1) + V(Y_2) + \ldots + V(Y_k)$$

We apply that result to get

$$V(X) = V(X_1) + V(X_2) + \ldots + V(X_n) = np(1 - p)$$

If you are not happy with the proof for variance, because it uses an unproved result, please go to page 107-108 of the text book, for a rigorous algebraic proof.

**CAUTION** We have used $V(Y_1 + Y_2 + \ldots + Y_k) = V(Y_1) + V(Y_2) + \ldots + V(Y_k)$ in the above proof, but take note that this equality does not always hold.

Homework :: 3.36b, 3.40, 3.41, 3.50, 3.54 a-b