BIVARIATE NORMAL ELLIPSES (Jennrich and Turner, 1969)

Suppose the observed locations are a random sample from a bivariate normal distribution with mean \((\mu_x, \mu_y)\) and variance-covariance matrix

\[
\Sigma = \begin{bmatrix}
\sigma_{xx} & \sigma_{xy} \\
\sigma_{yx} & \sigma_{yy}
\end{bmatrix}.
\]

\(\mu_x\) is the mean of the \(X\), \(\mu_y\) is the mean of the \(Y\), \(\sigma_{xx}\) is the variance of \(X\), \(\sigma_{yy}\) is the variance of \(Y\), and \(\sigma_{xy}\) is the covariance of \(X\) and \(Y\).

Use of a distribution implies differential utilization of space by an animal (or group). High values imply high use and low values uncommon use.

The first step is to estimate the parameters of the distribution; the theory for the bivariate Normal distribution has been well known for a long time, so the estimators are:

(from: http://www2.kenyon.edu/people/hartlaub/MellonProject/Bivariate2.html)
\[(\hat{\mu}_x, \hat{\mu}_y) = (\bar{x}, \bar{y})\]

\[\hat{\sigma}_{xx} = s_{x}^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2\]

\[\hat{\sigma}_{yy} = s_{y}^2 = \frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{y})^2\]

\[\hat{\sigma}_{xy} = s_{xy} = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})\]

Then,

\[\hat{\Sigma} = \begin{bmatrix} \hat{\sigma}_{xx} & \hat{\sigma}_{xy} \\ \hat{\sigma}_{yx} & \hat{\sigma}_{yy} \end{bmatrix}.\]

Now, by definition the range of the values from a normal distribution in any dimension is \((-\infty, +\infty)\), so we cannot calculate a home range for 100% coverage like an MCP. Instead, the two quantities of interest are a \((1-\alpha)100\%\) home range area estimate and the ellipse on the map indicating the plot of the \((1-\alpha)100\%\) home range.

The area estimate is given by

\[\hat{A}_{(1-\alpha)} = \pi |\hat{\Sigma}|^{1/2} \chi^2_{(1-\alpha, 2)}\]

where \(\chi^2_{(1-\alpha, 2)}\) is the cutoff value for a chi-square random variable on 2 degrees of freedom and with a tail area equal to \(\alpha\) and \(|\hat{\Sigma}|^{1/2}\) is the square root of the determinant of the 2x2 matrix \(\hat{\Sigma}\).

A 2x2 determinant is defined to be

\[\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = |a \ c \ b \ d| = ad - bc.\]
So, \[ \hat{A}_{(1-\alpha)} = \pi \left( \hat{\sigma}_{xx} \hat{\sigma}_{yy} - (\hat{\sigma}_{xy})^2 \right)^{1/2} \chi^{2}_{(1-\alpha, 2)}. \]

Some example values of the chi-square statistics are:

<table>
<thead>
<tr>
<th>(1-\alpha)100%</th>
<th>\chi^2_{(1-\alpha, 2)}</th>
</tr>
</thead>
<tbody>
<tr>
<td>99</td>
<td>9.21</td>
</tr>
<tr>
<td>95</td>
<td>5.99</td>
</tr>
<tr>
<td>90</td>
<td>4.61</td>
</tr>
<tr>
<td>75</td>
<td>2.77</td>
</tr>
<tr>
<td>60</td>
<td>1.83</td>
</tr>
<tr>
<td>50</td>
<td>1.38</td>
</tr>
</tbody>
</table>

The shape of the isopleth for the (1-\alpha)100% home range is an ellipse. The ellipse is centered on \((\hat{\mu}_x, \hat{\mu}_y) = (\bar{x}, \bar{y})\) and the line is drawn by plotting the following parametric equations:

\[
\begin{align*}
x &= \bar{x} + a \cos \psi \cos \theta - b \sin \psi \sin \theta \\
y &= \bar{y} + a \cos \psi \cos \theta - b \sin \psi \sin \theta
\end{align*}
\]

where

\[
\begin{align*}
R &= \sqrt{\left(\hat{\sigma}_{yy} - \hat{\sigma}_{xx}\right)^2 + 4(\hat{\sigma}_{xy})^2} \\
a &= \sqrt{\frac{\left(\hat{\sigma}_{yy} + \hat{\sigma}_{xx} + R\right) \chi^2_{(1-\alpha, 2)}}{2}} \\
b &= \sqrt{\frac{\left(\hat{\sigma}_{yy} + \hat{\sigma}_{xx} - R\right) \chi^2_{(1-\alpha, 2)}}{2}} \\
\theta &= \arctan\left(-\frac{2\hat{\sigma}_{xy}}{\hat{\sigma}_{yy} - \hat{\sigma}_{xx} - R}\right)
\end{align*}
\]
and $\psi$ is an angle that is varied in small increments around the compass (usually between 0 and 90°). Here $a$ and $b$ are the semi-axes and $\theta$ is the angle by which the major axis is inclined relative to the $X$-axis.

Advantages:
1) the area estimate is not a function of sample size, only its precision depends on $N$. In fact, the $(1-\alpha)100\%$ confidence interval for the area is

$$ \frac{(2N-4)\hat{A}}{\chi^2(1-\alpha/2,2N-4)} < \hat{A} < \frac{(2N-4)\hat{A}}{\chi^2(\alpha/2,2N-4)} $$

2) At least 100 observations are required in order to estimate a confidence interval of ± 20% (relative change).
3) The intractability to sample size allows comparability among studies with different sample sizes

Disadvantages:
1) the assumption of a single center
2) the assumption of a bivariate normal shape to the utilization of space
3) requires independence of the observations in order to calculate confidence intervals for area estimates or confidence bands around the plotted ellipses

Extensions:

Dunn and Rennolls (1983) attempted to modify this work to allow several “attraction points”, i.e. centers of high utilization such as dens or nests, by considering a set of K circular bivariate normal distributions (circular $\equiv \hat{\sigma}_{xx} = \hat{\sigma}_{yy}$). Each circular normal is centered over an attraction point and is allowed to have its own variance.

Dunn and Gipson (1977) considered the temporal non-independence of observations by assuming that the dependence can be described by the bivariate Ornstein-Uhlenbeck stochastic diffusion process. So, the autocorrelation of observations is accounted for in the estimation process but the estimators are still based on a model in which the home range is defined by a bivariate normal distribution.