Completely Randomized Designs

Completely randomized designs are used whenever the experimental units are *homogeneous*. This does not mean they are exactly alike. Rather, it means there is no identifiable source of variation in the EU’s. Treatments are assigned to EU’s *completely at random*. This means any EU has the same probability of receiving any treatment.
Concepts of Experimental Design

- Homogeneous EU
Concepts of Experimental Design

- Homogeneous EU
- Completely Randomize Treatments
Example of Completely Randomized Design

Wear due to friction applied to samples of wood veneer material

- Five brands of synthetic wood veneer material were compared for their durability.
- Four samples were used from each brand.
- The samples were subjected to a friction test in a randomly assigned order.
- Amounts of wear resulting from the friction test were measured on each sample.
- The random assignment of samples to the friction test avoids systematic bias that might result from the first to the last test.
### Data and Summary Statistics for Complete Randomized Design

<table>
<thead>
<tr>
<th>Brand</th>
<th>ACME</th>
<th>AJAX</th>
<th>CHAM</th>
<th>TUFF</th>
<th>XTRA</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.3</td>
<td>2.2</td>
<td>2.2</td>
<td>2.4</td>
<td>2.3</td>
<td></td>
</tr>
<tr>
<td>2.1</td>
<td>2.0</td>
<td>2.3</td>
<td>2.7</td>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td>2.4</td>
<td>1.9</td>
<td>2.4</td>
<td>2.6</td>
<td>2.3</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>2.1</td>
<td>2.6</td>
<td>2.7</td>
<td>2.4</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2.325</td>
<td>0.171</td>
<td>0.0292</td>
</tr>
<tr>
<td></td>
<td>2.050</td>
<td>0.129</td>
<td>0.0166</td>
</tr>
<tr>
<td></td>
<td>2.375</td>
<td>0.171</td>
<td>0.0292</td>
</tr>
<tr>
<td></td>
<td>2.600</td>
<td>0.141</td>
<td>0.0199</td>
</tr>
<tr>
<td></td>
<td>2.375</td>
<td>0.096</td>
<td>0.0092</td>
</tr>
</tbody>
</table>
Analysis of variance partitions variability in the data into sources of variation due to differences Between brands, and to differences Within brands. Results are usually presented in a table such as the following:

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between Brands</td>
<td>4</td>
<td>0.617</td>
<td>0.154</td>
<td>7.40</td>
</tr>
<tr>
<td>Within Brands</td>
<td>15</td>
<td>0.315</td>
<td>0.021</td>
<td></td>
</tr>
</tbody>
</table>
Interpretation of Mean Squares

• The “mean square” for “Between Brands,” equal to 0.154, is simply the computed variance of the brand means, multiplied by a constant. This is the summary measure of differences between the brands.

• The “mean square” for “Within Brands,” equal to 0.021, is the pooled variance within the brands. This is a summary measure of random variability within brands.
F-statistic from ANOVA

• The ratio of mean squares, $F=0.154/0.021=7.40$, reflects the magnitude of variation between brands relative to random variation. This says that the measure of variation between brands is 7.40 times larger than would be expected if the population means were equal.

• The statistical significance of an F ratio this large is $p=0.0017$. This is the probability of getting a value as large as 7.40 from an F distribution with 4 numerator degrees of freedom and 15 denominator degrees of freedom. Thus, we infer that there are statistically significant differences (at the $p=0.0017$ level) among the population means.
Analysis of variance computations

- Results of an analysis of variance are usually displayed in a tabular form such as:

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments</td>
<td>t - 1</td>
<td>SS(T)</td>
<td>MS(T)</td>
<td>MS(T)/MS(E)</td>
</tr>
<tr>
<td>Error</td>
<td>t(n -1)</td>
<td>SS(E)</td>
<td>MS(E)</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>tn – 1</td>
<td>SS(Tot)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- The term “treatment” is used generically to stand for a grouping variable, such as brands in the previous example.
- Likewise, “error” is a generic expression for random variation.
The primary computations are the *sums or squares*, SS(T) and SS(E). These quantities are computed from means and totals of data.

The sums of squares SS(T) and SS(E) form a partitioning of the total variation:

\[ SS(\text{Tot}) = SS(T) + SS(E) \]
Observed Data in Completely Randomized Design

\[ y_{ij} = \text{observation on jth EU assigned to treatment } i \]
Computations in ANOVA

\[ y_{ij} = \text{observation } j \text{ in group } i \]

\[ \bar{y}_{i.} = \text{mean of data in group } i \]

\[ y_{i.} = \text{total of data in group } i \]

\[ \bar{y}_{..} = \text{mean of all data} \]

\[ y_{..} = \text{total of all data} \]
Motivation of Sums of Squares

The partitioning of sums of squares corresponds to partitioning of differences:

Deviation of observation from overall mean
= (deviation of observation from group mean)
+ (deviation for group mean from overall mean)

\[ y_{ij} - \bar{y}_. = (y_{ij} - \bar{y}_i.) + (\bar{y}_i. - \bar{y}_.) \]
Sums of Squares

Definitional form:

\[ SS(\text{Tot}) = \sum_{ij} (y_{ij} - \bar{y}_{..})^2 \]

\[ SS(\text{T}) = \sum_{ij} (y_{ij} - \bar{y}_{i.})^2 \]

\[ SS(\text{E}) = n\sum_{i} (\bar{y}_{i.} - \bar{y}_{..})^2 \]
Sums of Squares

• The definitional form shows that:
  - $SS(T)$ measures variation of the group means about the overall mean
  - $SS(E)$ measures variation of data values about the group means
  - $SS(Tot)$ measures variation of individual data about the overall mean.
Degrees of Freedom

- The definitional form of SS also reveals the degrees of freedom for the sums of squares:

  - SS(T) is a sum of squared deviations of \( t \) numbers about their mean, so it has \( t-1 \) df
  - SS(E) is the sum of \( t \) SS terms, each of which has \( n-1 \) df, so the sum has \( t(n-1) \) df
  - SS(Tot) is the sum of squares of \( nt \) observations, and has \( nt-1 \) degrees of freedom.
Sums of Squares

Computational Form in terms of means:

\[
\begin{align*}
SS(\text{Tot}) &= \sum_{ij} y_{ij}^2 - nty_{..}^2 \\
SS(\text{T}) &= n\sum_i \bar{y}_i^2 - nty_{..}^2 \\
SS(\text{E}) &= \sum_{ij} y_{ij}^2 - \sum_i ny_i^2
\end{align*}
\]

Note: Each squared mean is multiplied by the number of observations in the mean
Sums of Squares

Computational Form in terms of totals:

\[
SS(\text{Tot}) = \frac{\sum_{ij} y_{ij}^2 - y_{..}^2}{nt} \\
SS(\text{T}) = \frac{\sum_i y_{i.}^2}{n} - \frac{y_{..}^2}{nt} \\
SS(\text{E}) = \frac{\sum_{ij} y_{ij}^2 - \sum_i y_{i.}^2}{n}
\]

Note: Each squared total is divided by the number of observations in the total.
Partitioning of SS and df

- Using either of the computational forms, it is easy to see that
  - \( SS(\text{Tot}) = SS(T) + SS(E) \)

- Also,
  - \( df(\text{Tot}) = df(T) + df(E) \)
Muzzle Velocity Experiment

- Rifle cartridges were made using three types of gunpowder.
- Four cartridges of each powder type were fired from a rifle in random order and muzzle velocities were measured.
# Muzzle Velocities of bullets made from three types of Gunpowder

<table>
<thead>
<tr>
<th>Powder Type</th>
<th>Muzzle Velocities</th>
<th>Totals</th>
<th>Means</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>27.1  28.1  27.4  27.7  28.0  28.1  27.4  27.1</td>
<td>220.9</td>
<td>27.6125</td>
</tr>
<tr>
<td>B</td>
<td>28.3  27.9  28.1  28.3  27.9  27.6  28.5  27.9</td>
<td>224.5</td>
<td>28.0625</td>
</tr>
<tr>
<td>C</td>
<td>28.4  28.9  28.3  27.9  28.2  28.9  28.8  27.7</td>
<td>227.1</td>
<td>28.3875</td>
</tr>
<tr>
<td>Combined</td>
<td>24</td>
<td>672.5</td>
<td>28.0208</td>
</tr>
</tbody>
</table>

**Legend:**
- **n**: Number of observations
- **Totals**: Sum of muzzle velocities
- **Means**: Mean muzzle velocity
Sum of Squares Between Powders (SST)

Definitional:
8(27.61 – 28.02)^2 + 8(28.06 – 28.02)^2 + 8(28.38 – 28.02)^2
= 2.42

Computational:
= 220.9^2/8 + 224.5^2/8 + 227.1^2/8 – 672.5^2/24
= 2.42
Sum of Squares Within Powders (SSE)

**Definitional:**

\[(27.1 - 27.6125)^2 + \ldots + (27.1 - 27.6125)^2\]
\[+ (28.3 - 28.0625)^2 + \ldots + (27.9 - 28.0625)^2\]
\[+ (28.4 - 28.3875)^2 + \ldots + (27.7 - 28.3875)^2\]

**Computational:**

\[27.1^2 + 28.1^2 + \ldots + 27.1^2 - 220.9^2/8\]
\[+ 28.3^2 + 27.9^2 + \ldots + 27.9^2 - 224.5^2/8\]
\[+ 28.4^2 + 28.9^2 + \ldots + 27.7^2 - 227.1^2/8\]

\[= 3.30\]
Sum of Squares Total (SS(Tot))

Definitional:
\[(27.1 - 28.02)^2 + \ldots + (27.7 - 28.02)^2\]

Computational:
\[27.1^2 + 28.1^2 + 27.7^2 - \frac{672.5^2}{24}\] = 5.72
### Analysis of Variance Table

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between</td>
<td>2</td>
<td>2.42</td>
<td>1.21</td>
<td>7.72</td>
<td>0.0031</td>
</tr>
<tr>
<td>Within</td>
<td>21</td>
<td>3.30</td>
<td>0.17</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>23</td>
<td>5.72</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Conclusion: The mean muzzle velocities are statistically different at the significance level $p=0.0031$. 
Model for one-way classification

- Let $\mu_i$ denote the population mean for the $i$th group.
- An observation $y_{ij}$ from this population can be expressed as the population mean pulse a random deviation from the mean

$$y_{ij} = \mu_i + \varepsilon_{ij}$$
Parameters in Model

In addition, the population mean for the $i^{th}$ group is:

$$\mu_i = \mu + \alpha_i$$

where the overall mean is:

$$\mu = \frac{\sum \mu_i}{t}$$

and the group effect is:

$$\alpha_i = \mu_i - \mu$$
Putting the parts together gives:

\[ y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \]
Model Assumptions

The random deviations $\varepsilon_{ij}$ are assumed independent and normally distributed with mean 0 and variance $\sigma^2$. 
Motivation for the F test

• The null hypothesis is
  \[ H_0: \mu_1 = \mu_2 = \ldots = \mu_t \]

• It can be shown that:
  MS(T) estimates \( \sigma^2 + n\sum_i(\mu_i - \mu)^2 \)
  and
  MS(E) estimates \( \sigma^2 \)
Motivation for the F test (cont’d)

• The quantity \( n\sum_i(\mu_i - \mu)^2 \) would be equal to 0 if \( H_0: \mu_1 = \mu_2 = \ldots = \mu_t \) is true.

• Thus, if \( H_0: \mu_1 = \mu_2 = \ldots = \mu_t \) is true, then \( MS(T) \) would estimate \( \sigma^2 \).

• Recall: \( MS(E) \) estimates \( \sigma^2 \)
Motivation for the F test (cont’d)

- The ratio $F = \frac{MS(\text{Trt})}{MS(\text{Error})}$ gives an indication of whether $H_0$ is true or false.
  
  - If $F$ is large, this indicates $H_0$ is false.
  
  - If $F$ is small (around 1) this indicates $H_0$ is true.