STA6329 Exercise 7 **Solutions to ex1-ex4**

1. Let $\boldsymbol{A}$ represent a symmetric matrix and k an integer greater than or equal to 1. Show that if $\boldsymbol{A}^{k+1} = \boldsymbol{A}^k$, then $\boldsymbol{A}$ is idempotent.

   Solution: Note that if $\boldsymbol{A}^{k+1} = \boldsymbol{A}^k$, and $\boldsymbol{A}$ is symmetric then

$$\begin{aligned} \boldsymbol{A}^{k+1} &= \boldsymbol{A}^k \\ \boldsymbol{A}^2\boldsymbol{A}^{k-1} &= \boldsymbol{A}^2\boldsymbol{A}^{k-2} \\ \boldsymbol{A}'\boldsymbol{A}\boldsymbol{A}^{k-1} &= \boldsymbol{A}'\boldsymbol{A}\boldsymbol{A}^{k-2} \\ \boldsymbol{A}\boldsymbol{A}^{k-1} &= \boldsymbol{A}\boldsymbol{A}^{k-2} \end{aligned}$$

   The last equality holds by Corollary 5.3.3. Thus if $\boldsymbol{A}^{k+1} = \boldsymbol{A}^k$ then $\boldsymbol{A}^k = \boldsymbol{A}^{k-1}$. Repeating the above argument another $k-2$ times, we arrive at $\boldsymbol{A}^2 = \boldsymbol{A}$ and thus $\boldsymbol{A}$ is idempotent.

2. Let $\boldsymbol{A}$ represent an $r \times m$ matrix and $\boldsymbol{B}$ an $m \times n$ matrix.

   (a) Show that $\boldsymbol{B}^-\boldsymbol{A}^-$ is a generalized inverse of $\boldsymbol{AB}$ if and only if $\boldsymbol{A}^-\boldsymbol{ABB}^-$ is idempotent.

   Solution: Assume that $\boldsymbol{B}^-\boldsymbol{A}^-$ is the generalized inverse of $\boldsymbol{AB}$, then

$$\boldsymbol{AB}(\boldsymbol{B}^-\boldsymbol{A}^-)\boldsymbol{AB} = \boldsymbol{AB}$$

   Pre-multiplying both sides by $\boldsymbol{A}^-$ and post-multiplying both sides by $\boldsymbol{B}^-$ gives

$$\begin{aligned} \boldsymbol{A}^-\boldsymbol{ABB}^-\boldsymbol{A}^-\boldsymbol{ABB}^- &= \boldsymbol{A}^-\boldsymbol{ABB}^- \\ (\boldsymbol{A}^-\boldsymbol{ABB}^-)^2 &= \boldsymbol{A}^-\boldsymbol{ABB}^- \end{aligned}$$

   Thus $\boldsymbol{A}^-\boldsymbol{ABB}^-$ is idempotent.

   Next assume that $\boldsymbol{A}^-\boldsymbol{ABB}^-$ is idempotent, then

$$\boldsymbol{A}^-\boldsymbol{ABB}^-\boldsymbol{A}^-\boldsymbol{ABB}^- = \boldsymbol{A}^-\boldsymbol{ABB}^-$$

   Pre-multiplying both sides by $\boldsymbol{A}$ and post-multiplying both sides by $\boldsymbol{B}$ gives

$$\begin{aligned} \boldsymbol{AA}^-\boldsymbol{ABB}^-\boldsymbol{A}^-\boldsymbol{ABB}^-\boldsymbol{B} &= \boldsymbol{AA}^-\boldsymbol{ABB}^-\boldsymbol{B} \\ \boldsymbol{ABB}^-\boldsymbol{A}^-\boldsymbol{AB} &= \boldsymbol{AB} \end{aligned}$$

   Thus $\boldsymbol{B}^-\boldsymbol{A}^-$ is the generalized inverse of $\boldsymbol{AB}$. Hence $\boldsymbol{B}^-\boldsymbol{A}^-$ is a generalized inverse of $\boldsymbol{AB}$ if and only if $\boldsymbol{A}^-\boldsymbol{ABB}^-$ is idempotent.

(b) Show that if $\boldsymbol{A}$ has full column rank or if $\boldsymbol{B}$ has full row rank, then $\boldsymbol{B}^{-}\boldsymbol{A}^{-}$ is then generalized inverse of $\boldsymbol{AB}$.

Solution: Suppose that $\boldsymbol{A}$ has full column rank. From Lemma 9.2.8, $\boldsymbol{A}^{-}$ is the left inverse of $\boldsymbol{A}$. Thus

$$\begin{aligned}\boldsymbol{AB}(\boldsymbol{B}^{-}\boldsymbol{A}^{-})\boldsymbol{AB} &= \boldsymbol{ABB}^{-}(\boldsymbol{A}^{-}\boldsymbol{A})\boldsymbol{B}\\ &= \boldsymbol{ABB}^{-}\boldsymbol{IB}\\ &= \boldsymbol{ABB}^{-}\boldsymbol{B}\\ &= \boldsymbol{AB}\end{aligned}$$

Thus $\boldsymbol{B}^{-}\boldsymbol{A}^{-}$ is the generalized inverse of $\boldsymbol{A}$.

Next suppose that $\boldsymbol{B}$ has full row rank. From Lemma 9.2.8, $\boldsymbol{B}^{-}$ is the right inverse of $\boldsymbol{B}$. Hence

$$\begin{aligned}\boldsymbol{AB}(\boldsymbol{B}^{-}\boldsymbol{A}^{-})\boldsymbol{AB} &= \boldsymbol{A}(\boldsymbol{BB}^{-})\boldsymbol{A}^{-}\boldsymbol{AB}\\ &= \boldsymbol{AIA}^{-}\boldsymbol{AB}\\ &= \boldsymbol{AA}^{-}\boldsymbol{AB}\\ &= \boldsymbol{AB}\end{aligned}$$

Again, we obtain that $\boldsymbol{B}^{-}\boldsymbol{A}^{-}$ is the generalized inverse of $\boldsymbol{AB}$.

3. Let $\boldsymbol{A}$ be an $n \times n$ matrix and $\boldsymbol{x}$ be an $n \times 1$ vector. Show that

(i) If $\boldsymbol{Ax} = 0$ for all $\boldsymbol{x}$, then $\boldsymbol{A} = 0$.

Solution: Define the $n \times 1$ vector $\boldsymbol{e}_i$ such that $e_j = 1$ if $j = i$ and $e_j = 0$ for $j \neq i$. Thus

$$\boldsymbol{Ae}_i = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \ldots & \boldsymbol{a}_n \end{bmatrix} \boldsymbol{e}_i = \boldsymbol{a}_i$$

for $i = 1, \ldots, n$. Since $\boldsymbol{Ax} = 0$ for all $\boldsymbol{x}$ it follows that $\boldsymbol{a}_i = 0$ for $i = 1, \ldots, n$. Therefore

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \ldots & \boldsymbol{0} \end{bmatrix} = \boldsymbol{0}$$

(2) If $\boldsymbol{A}$ is symmetric and $\boldsymbol{x'Ax} = 0$ for all $\boldsymbol{x}$, then $bmA = 0$.

Solution: Define the vector $\boldsymbol{e}_i$ as above. Then for $\boldsymbol{A} = \{a_{ij}\}$, $i, j \in \{1, \ldots, n\}$

$$\boldsymbol{e}_i'\boldsymbol{Ae}_i = a_{ii} = 0$$

Thus all diagonal elements of $\boldsymbol{A}$ are zero.

Next define the $n \times 1$ vector $\boldsymbol{e}_{ij}$ ($i \neq j$ and $i, j = 0, \ldots, n$) such that $e_k = 1$ if $k = i, j$ and $e_k = 0$ for $k \neq i, j$. Then

$$\begin{aligned}
\boldsymbol{e}'_{ij}\boldsymbol{A}\boldsymbol{e}_{ij} &= \boldsymbol{e}'_{ij} \left[\begin{array}{cccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \ldots & \boldsymbol{a}_n \end{array} \right] \boldsymbol{e}_{ij} \\
&= \boldsymbol{e}'_{ij} \left[\boldsymbol{a}_i + \boldsymbol{a}_j\right] \\
&= (a_{ii} + a_{ij}) + (a_{ji} + a_{jj}) \\
&= a_{ij} + a_{ji} = 2a_{ij}
\end{aligned}$$

Since $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} = 0$ for all $\boldsymbol{x}$, $a_{ij} = 0$ for all $i, j = 1, \ldots, n$. Therefore $\boldsymbol{A} = \boldsymbol{0}$.

(iii) If $\boldsymbol{A}$ is not symmetric, then $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} = 0$ for all $\boldsymbol{x}$ implies $\boldsymbol{A} = -\boldsymbol{A}'$.

From (ii), if $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} = 0$ for all $\boldsymbol{x}$ then $a_{ii} = 0$ for $i = 1, \ldots, n$ and for $i \neq j$,

$$\boldsymbol{e}'_{ij}\boldsymbol{A}\boldsymbol{e}_{ij} = a_{ij} + a_{ji} = 0$$

Thus a_{ij} must equal $-a_{ji}$ ($i \neq j$) for the above equality to hold. Since the diagonal elements are zero, $a_{ij} = -a_{ji}$ for all $i, j \in \{1, \ldots, n\}$. Hence $\boldsymbol{A} = -\boldsymbol{A}$.

4. We have shown in class that the projection matrix $\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'$ is symmetric and idempotent. show that any $\boldsymbol{A}_{n\times n}$ is a projection matrix if and only if it is symmetric and idempotent.

Solution: Suppose that $\boldsymbol{A}$ is a projection matrix. By Corollary 12.3.3, $\boldsymbol{A} = \boldsymbol{P}_{\boldsymbol{X}} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'$ for some matrix $\boldsymbol{X}$. We have already shown that $\boldsymbol{P}_{\boldsymbol{X}}$ is symmetric and idempotent, thus $\boldsymbol{A}$ is symmetric and idempotent.

Next suppose that $\boldsymbol{A}$ is symmetric and idempotent. Since $\boldsymbol{A}$ is idempotent, $\boldsymbol{A}^{-} = (\boldsymbol{A}^2)^{-}$. Thus

$$\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-}\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{A}^2)^{-}\boldsymbol{A}$$

Using the fact that $\boldsymbol{A}$ is symmetric,

$$\boldsymbol{A}(\boldsymbol{A}^2)^{-}\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^{-}\boldsymbol{A}'$$

Thus $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^{-}\boldsymbol{A}' = \boldsymbol{P}_{\boldsymbol{A}}$ and, by Corollary 12.2.3, $\boldsymbol{A}$ is a projection matrix. Therefore $\boldsymbol{A}_{n\times n}$ is a projection matrix if and only if it is symmetric and idempotent.