1. Let $\boldsymbol{A}$ represent an $m \times n$ matrix. Show that any $n \times m$ matrix $\boldsymbol{X}$ such that $\boldsymbol{A'AX} = \boldsymbol{A'}$ is a generalized inverse of $\boldsymbol{A}$ and similarly that any $n \times m$ matrix $\boldsymbol{Y}$ such that $\boldsymbol{AA'Y'} = \boldsymbol{A}$ is a generalized inverse of $\boldsymbol{A}$.

   Proof: $\boldsymbol{A'AX} = \boldsymbol{A'} \Rightarrow \boldsymbol{A'AXA} = \boldsymbol{A'A} \Rightarrow \boldsymbol{AXA} = \boldsymbol{A}$ by a problem in midterm 1 that $\boldsymbol{(A'A)^{-}A'}$ is a generalized inverse of A. The second half of this problem is similar.

2. Let $\boldsymbol{A}$ represent an $m \times n$ nonnull matrix, let $\boldsymbol{B}$ represent a matrix of full column rank and $\boldsymbol{T}$ a matrix of full row rank such that $\boldsymbol{A} = \boldsymbol{BT}$, and let $\boldsymbol{L}$ represent a left inverse of $\boldsymbol{B}$ and $\boldsymbol{R}$ a right inverse of $\boldsymbol{T}$

   (a) Show that the matrix $\boldsymbol{R(B'B)^{-1}R'}$ is a generalized inverse of the matrix $\boldsymbol{A'A}$ and that $\boldsymbol{L'(T'T)^{-1}L}$ is a generalized inverse of $\boldsymbol{AA'}$.

   Proof:

$$\begin{aligned}\boldsymbol{A'A(R(B'B)^{-1}R')A'A} &= \boldsymbol{T'B'BTR(B'B)^{-1}R'T'B'BT} \\ &= \boldsymbol{T'B'B(B'B)^{-1}B'BT} \\ &= \boldsymbol{T'B'BT} = \boldsymbol{A'A}\end{aligned}$$

   Thus $\boldsymbol{R(B'B)^{-1}R'}$ is a generalized inverse of the matrix $\boldsymbol{A'A}$. Next,

$$\begin{aligned}\boldsymbol{AA'(L'(T'T)^{-1}L)AA'} &= \boldsymbol{BTT'B'L'(T'T)^{-1}LBTT'B'} \\ &= \boldsymbol{BTT'(T'T)^{-1}TT'B'} \\ &= \boldsymbol{BTT'B'} = \boldsymbol{AA'}\end{aligned}$$

   Therefore $\boldsymbol{L'(T'T)^{-1}L}$ is a generalized inverse of $\boldsymbol{AA'}$.

   (b) Show that if $\boldsymbol{A}$ is symmetric, then the matrix $\boldsymbol{R(TB)^{-1}L}$ is a generalized inverse of the matrix $\boldsymbol{A}^2$.

Proof: Since $\boldsymbol{A}$ is symmetric, $\boldsymbol{TB}$ is non-singular and

$$\boldsymbol{A}^2\boldsymbol{R}(\boldsymbol{TB})^{-1}\boldsymbol{L}\boldsymbol{A}^2 = \boldsymbol{AAR}(\boldsymbol{TB})^{-1}\boldsymbol{LAA}$$

$$= \boldsymbol{BTBTR}(\boldsymbol{TB})^{-1}\boldsymbol{LBTBT}$$

$$= \boldsymbol{BTB}(\boldsymbol{TB})^{-1}\boldsymbol{TBT}$$

$$= \boldsymbol{BTBT} = \boldsymbol{A}^2$$

Thus $\boldsymbol{R}(\boldsymbol{TB})^{-1}\boldsymbol{L}$ is a generalized inverse of $\boldsymbol{A}^2$.

3. Let $\boldsymbol{A}$ represent an $m \times n$ nonnull matrix of rank r. Take $\boldsymbol{B}$ and $\boldsymbol{K}$ to be nonsingular matrices such that

$$\boldsymbol{A} = \boldsymbol{B}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{K}$$

(a) Show that an $n \times m$ matrix $\boldsymbol{G}$ is a generalized inverse if and only if $\boldsymbol{G}$ is expressible in the form

$$\boldsymbol{G} = \boldsymbol{K}^{-1}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{U} \\ \boldsymbol{V} & \boldsymbol{W} \end{bmatrix}\boldsymbol{B}^{-1}$$

for some $r \times (m-r)$ matrix $\boldsymbol{U}$, $(n-r) \times r$ matrix $\boldsymbol{V}$ and $(n-r) \times (m-r)$ matrix $\boldsymbol{W}$.

Proof: Assume that $\boldsymbol{G}$ has the form

$$\boldsymbol{G} = \boldsymbol{K}^{-1}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{U} \\ \boldsymbol{V} & \boldsymbol{W} \end{bmatrix}\boldsymbol{B}^{-1}$$

Then

$$\boldsymbol{AGA} = \boldsymbol{B}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{KK}^{-1}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{U} \\ \boldsymbol{V} & \boldsymbol{W} \end{bmatrix}\boldsymbol{B}^{-1}\boldsymbol{B}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{K}$$

$$= \boldsymbol{B}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{U} \\ \boldsymbol{V} & \boldsymbol{W} \end{bmatrix}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{K}$$

$$= \boldsymbol{B}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{V} & \boldsymbol{0} \end{bmatrix}\boldsymbol{K}$$

$$= \boldsymbol{B}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{K}$$

$$= \boldsymbol{A}$$

Therefore $\boldsymbol{G}$ is a generalized inverse of $\boldsymbol{A}$. Next assume that $\boldsymbol{G}$ is the generalized inverse of $\boldsymbol{A}$. Then

$$\boldsymbol{AGA} = \boldsymbol{B}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{KGB}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{K} = \boldsymbol{B}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{K}$$

Let $\boldsymbol{P} = \boldsymbol{KGB}$ where

$$P = \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix}$$

where $\boldsymbol{P}_{12}$ is an $r\times(m-r)$ matrix, $\boldsymbol{P}_{21}$ is an $(n-r)\times r$ matrix and $\boldsymbol{P}_{22}$ is an $(n-r)\times(m-r)$ matrix. It follows that

$$\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

if and only if $\boldsymbol{P}_{11} = \boldsymbol{I}_r$. Thus

$$\boldsymbol{KGB} = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix}$$

and therefore

$$G = \boldsymbol{K}^{-1}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix}\boldsymbol{B}^{-1}$$

which has the form

$$\boldsymbol{G} = \boldsymbol{K}^{-1}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{U} \\ \boldsymbol{V} & \boldsymbol{W} \end{bmatrix}\boldsymbol{B}^{-1}$$

Hence $\boldsymbol{G}$ is the generalized inverse of $\boldsymbol{A}$ if and only if it is expressible in the above form.

(b) Show that distinct choices of $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{W}$ lead to distinct generalized inverses.

Proof: Let

$$\boldsymbol{G}_1 = \boldsymbol{K}^{-1}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{U}_1 \\ \boldsymbol{V}_1 & \boldsymbol{W}_1 \end{bmatrix}\boldsymbol{B}^{-1} \quad \text{and} \quad \boldsymbol{G}_2 = \boldsymbol{K}^{-1}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{U}_2 \\ \boldsymbol{V}_2 & \boldsymbol{W}_2 \end{bmatrix}\boldsymbol{B}^{-1}$$

Assume that $\boldsymbol{G}_1 = \boldsymbol{G}_2$, then

$$\boldsymbol{K}^{-1}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{U}_1 \\ \boldsymbol{V}_1 & \boldsymbol{W}_1 \end{bmatrix}\boldsymbol{B}^{-1} = \boldsymbol{K}^{-1}\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{U}_2 \\ \boldsymbol{V}_2 & \boldsymbol{W}_2 \end{bmatrix}\boldsymbol{B}^{-1}$$

Pre-multiplying by $\boldsymbol{K}$ and post-multiplying by $\boldsymbol{B}$ gives

$$\begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{U}_1 \\ \boldsymbol{V}_1 & \boldsymbol{W}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{U}_2 \\ \boldsymbol{V}_2 & \boldsymbol{W}_2 \end{bmatrix}$$

which is true only if $\boldsymbol{U}_1 = \boldsymbol{U}_2$, $\boldsymbol{V}_1 = \boldsymbol{V}_2$ and $\boldsymbol{W}_1 = \boldsymbol{W}_2$.

4. Let $\boldsymbol{T}$ represent an $m \times p$ matrix, $\boldsymbol{U}$ an $m \times q$ matrix, $\boldsymbol{V}$ an $n \times p$ matrix and $\boldsymbol{W}$ an $n \times q$ matrix, take

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{T} & \boldsymbol{U} \\ \boldsymbol{V} & \boldsymbol{W} \end{bmatrix}$$

and define $\boldsymbol{Q} = \boldsymbol{W} - \boldsymbol{V}\boldsymbol{T}^{-}\boldsymbol{U}$

(a) Show that the matrix

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{T}^{-} + \boldsymbol{T}^{-}\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V}\boldsymbol{T}^{-} & -\boldsymbol{T}^{-}\boldsymbol{U}\boldsymbol{Q}^{-} \\ -\boldsymbol{Q}^{-}\boldsymbol{V}\boldsymbol{T}^{-} & \boldsymbol{Q} \end{bmatrix}$$

is a generalized inverse of $\boldsymbol{A}$ if and only if

(1) $(\boldsymbol{I} - \boldsymbol{T}\boldsymbol{T}^{-})\boldsymbol{U}(I - \boldsymbol{Q}^{-}\boldsymbol{Q}) = \boldsymbol{0}$,

(2) $(\boldsymbol{I} - \boldsymbol{Q}\boldsymbol{Q}^{-})\boldsymbol{V}(I - \boldsymbol{T}^{-}\boldsymbol{T}) = \boldsymbol{0}$, and

(3) $(\boldsymbol{I} - \boldsymbol{T}\boldsymbol{T}^{-})\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V}(I - \boldsymbol{T}^{-}\boldsymbol{T}) = \boldsymbol{0}$

Proof:

$$\boldsymbol{AGA} = \begin{bmatrix} \boldsymbol{T} & \boldsymbol{U} \\ \boldsymbol{V} & \boldsymbol{W} \end{bmatrix} \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{G}_{12} \\ \boldsymbol{G}_{21} & \boldsymbol{G}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{T} & \boldsymbol{U} \\ \boldsymbol{V} & \boldsymbol{W} \end{bmatrix}$$

Then

$$\boldsymbol{AGA} = \begin{bmatrix} \boldsymbol{TG}_{11}\boldsymbol{T} + \boldsymbol{TG}_{12}\boldsymbol{V} + \boldsymbol{UG}_{21}\boldsymbol{T} + \boldsymbol{UG}_{22}\boldsymbol{V} & \boldsymbol{TG}_{11}\boldsymbol{U} + \boldsymbol{TG}_{12}\boldsymbol{W} + \boldsymbol{UG}_{21}\boldsymbol{U} + \boldsymbol{UG}_{22}\boldsymbol{W} \\ \boldsymbol{VG}_{11}\boldsymbol{T} + \boldsymbol{VG}_{12}\boldsymbol{V} + \boldsymbol{WG}_{21}\boldsymbol{T} + \boldsymbol{WG}_{22}\boldsymbol{V} & \boldsymbol{VG}_{11}\boldsymbol{U} + \boldsymbol{VG}_{12}\boldsymbol{W} + \boldsymbol{WG}_{21}\boldsymbol{U} + \boldsymbol{WG}_{22}\boldsymbol{W} \end{bmatrix}$$

Looking at the first entry of $\boldsymbol{G}$,

$$\begin{aligned} &\boldsymbol{TG}_{11}\boldsymbol{T} + \boldsymbol{TG}_{12}\boldsymbol{V} + \boldsymbol{UG}_{21}\boldsymbol{T} + \boldsymbol{UG}_{22}\boldsymbol{V} \\ &\quad = \boldsymbol{T}(\boldsymbol{T}^{-} + \boldsymbol{T}^{-}\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V}\boldsymbol{T}^{-})\boldsymbol{T} + \boldsymbol{T}(-\boldsymbol{T}^{-}\boldsymbol{U}\boldsymbol{Q}^{-})\boldsymbol{V} + \boldsymbol{U}(-\boldsymbol{Q}^{-}\boldsymbol{V}\boldsymbol{T}^{-})\boldsymbol{T} + \boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V} \\ &\quad = \boldsymbol{TT}^{-}\boldsymbol{T} + \boldsymbol{TT}^{-}\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V}\boldsymbol{T}^{-}\boldsymbol{T} - \boldsymbol{TT}^{-}\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V} - \boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V}\boldsymbol{T}^{-}\boldsymbol{T} + \boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V} \\ &\quad = \boldsymbol{T} + (\boldsymbol{I} - \boldsymbol{TT}^{-})\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V} - (\boldsymbol{I} - \boldsymbol{TT}^{-})\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V}\boldsymbol{T}^{-}\boldsymbol{T} \\ &\quad = \boldsymbol{T} + (\boldsymbol{I} - \boldsymbol{TT}^{-})(\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V} - \boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V}\boldsymbol{T}^{-}\boldsymbol{T}) \\ &\quad = \boldsymbol{T} + (\boldsymbol{I} - \boldsymbol{TT}^{-})\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V}(\boldsymbol{I} - \boldsymbol{TT}^{-}) \end{aligned}$$

Next,

$$\begin{aligned} &\boldsymbol{TG}_{11}\boldsymbol{U} + \boldsymbol{TG}_{12}\boldsymbol{W} + \boldsymbol{UG}_{21}\boldsymbol{U} + \boldsymbol{UG}_{22}\boldsymbol{W} \\ &\quad = \boldsymbol{T}(\boldsymbol{T}^{-} + \boldsymbol{T}^{-}\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V}\boldsymbol{T}^{-})\boldsymbol{U} + \boldsymbol{T}(-\boldsymbol{T}^{-}\boldsymbol{U}\boldsymbol{Q}^{-})\boldsymbol{W} + \boldsymbol{U}(-\boldsymbol{Q}^{-}\boldsymbol{V}\boldsymbol{T}^{-})\boldsymbol{U} + \boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{W} \\ &\quad = \boldsymbol{TT}^{-}\boldsymbol{U} + \boldsymbol{TT}^{-}\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V}\boldsymbol{T}^{-}\boldsymbol{U} - \boldsymbol{TT}^{-}\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{W} - \boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V}\boldsymbol{T}^{-}\boldsymbol{U} + \boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{W} \\ &\quad = \boldsymbol{TT}^{-}\boldsymbol{U} + (\boldsymbol{I} - \boldsymbol{TT}^{-})\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{W} - (\boldsymbol{I} - \boldsymbol{TT}^{-})\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{V}\boldsymbol{T}^{-}\boldsymbol{U} \\ &\quad = \boldsymbol{TT}^{-}\boldsymbol{U} + (\boldsymbol{I} - \boldsymbol{TT}^{-})\boldsymbol{U}\boldsymbol{Q}^{-}(\boldsymbol{W} - \boldsymbol{V}\boldsymbol{T}^{-}\boldsymbol{U}) \\ &\quad = \boldsymbol{TT}^{-}\boldsymbol{U} + (\boldsymbol{I} - \boldsymbol{TT}^{-})\boldsymbol{U}\boldsymbol{Q}^{-}(\boldsymbol{Q} + \boldsymbol{V}\boldsymbol{T}^{-}\boldsymbol{U} - \boldsymbol{V}\boldsymbol{T}^{-}\boldsymbol{U}) \\ &\quad = \boldsymbol{TT}^{-}\boldsymbol{U} + (\boldsymbol{I} - \boldsymbol{TT}^{-})\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{Q} + (\boldsymbol{U} - \boldsymbol{U}) \\ &\quad = \boldsymbol{U} + (\boldsymbol{I} - \boldsymbol{TT}^{-})\boldsymbol{U}\boldsymbol{Q}^{-}\boldsymbol{Q} - (\boldsymbol{I} - \boldsymbol{TT}^{-})\boldsymbol{U} \\ &\quad = \boldsymbol{U} + (\boldsymbol{I} - \boldsymbol{TT}^{-})\boldsymbol{U}(\boldsymbol{Q}\boldsymbol{Q}^{-} - \boldsymbol{I}) \\ &\quad = \boldsymbol{U} - (\boldsymbol{I} - \boldsymbol{TT}^{-})\boldsymbol{U}(\boldsymbol{I} - \boldsymbol{Q}\boldsymbol{Q}^{-}) \end{aligned}$$

Similarly

$$
\begin{aligned}
&\boldsymbol{VG}_{11}\boldsymbol{T}+\boldsymbol{VG}_{12}\boldsymbol{V}+\boldsymbol{WG}_{21}\boldsymbol{T}+\boldsymbol{WG}_{22}\boldsymbol{V}\\
&\quad=\boldsymbol{V}(\boldsymbol{T}^-+\boldsymbol{T}^-\boldsymbol{UQ}^-\boldsymbol{VT}^-)\boldsymbol{T}+\boldsymbol{V}(-\boldsymbol{T}^-\boldsymbol{UQ}^-)\boldsymbol{V}+\boldsymbol{W}(-\boldsymbol{Q}^-\boldsymbol{VT}^-)\boldsymbol{T}+\boldsymbol{WQ}^-\boldsymbol{V}\\
&\quad=\boldsymbol{VT}^-\boldsymbol{T}+\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{VT}^-\boldsymbol{T}-\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{V}-\boldsymbol{WQ}^-\boldsymbol{VT}^-\boldsymbol{T}+\boldsymbol{WQ}^-\boldsymbol{V}\\
&\quad=\boldsymbol{VT}^-\boldsymbol{T}+\boldsymbol{WQ}^-\boldsymbol{V}(\boldsymbol{I}-\boldsymbol{TT}^-)-\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{V}(\boldsymbol{I}-\boldsymbol{TT}^-)\\
&\quad=\boldsymbol{VT}^-\boldsymbol{T}+(\boldsymbol{WQ}^-\boldsymbol{V}-\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{V})(\boldsymbol{I}-\boldsymbol{TT}^-)\\
&\quad=\boldsymbol{VT}^-\boldsymbol{T}+((\boldsymbol{Q}+\boldsymbol{VT}^-\boldsymbol{U})\boldsymbol{Q}^-\boldsymbol{V}-\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{V})(\boldsymbol{I}-\boldsymbol{TT}^-)\\
&\quad=\boldsymbol{VT}^-\boldsymbol{T}+(\boldsymbol{QQ}^-\boldsymbol{V}+\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{V}-\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{V})(\boldsymbol{I}-\boldsymbol{TT}^-)\\
&\quad=\boldsymbol{VT}^-\boldsymbol{T}+\boldsymbol{QQ}^-\boldsymbol{V}(\boldsymbol{I}-\boldsymbol{TT}^-)+(\boldsymbol{V}-\boldsymbol{V})\\
&\quad=\boldsymbol{V}+\boldsymbol{QQ}^-\boldsymbol{V}(\boldsymbol{I}-\boldsymbol{TT}^-)-\boldsymbol{V}(\boldsymbol{I}-\boldsymbol{TT}^-)\\
&\quad=\boldsymbol{V}+(\boldsymbol{QQ}^--\boldsymbol{I})\boldsymbol{V}(\boldsymbol{I}-\boldsymbol{TT}^-)\\
&\quad=\boldsymbol{V}-(\boldsymbol{I}-\boldsymbol{QQ}^-)\boldsymbol{V}(\boldsymbol{I}-\boldsymbol{TT}^-)
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\boldsymbol{VG}_{11}\boldsymbol{U}+\boldsymbol{VG}_{12}\boldsymbol{W}+\boldsymbol{WG}_{21}\boldsymbol{U}+\boldsymbol{WG}_{22}\boldsymbol{W}\\
&\quad=\boldsymbol{V}(\boldsymbol{T}^-+\boldsymbol{T}^-\boldsymbol{UQ}^-\boldsymbol{VT}^-)\boldsymbol{U}+\boldsymbol{V}(-\boldsymbol{T}^-\boldsymbol{UQ}^-)\boldsymbol{W}+\boldsymbol{W}(-\boldsymbol{Q}^-\boldsymbol{VT}^-)\boldsymbol{U}+\boldsymbol{WQ}^-\boldsymbol{W}\\
&\quad=\boldsymbol{VT}^-\boldsymbol{U}+\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{VT}^-\boldsymbol{U}-\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{W}-\boldsymbol{WQ}^-\boldsymbol{VT}^-\boldsymbol{U}+\boldsymbol{WQ}^-\boldsymbol{W}\\
&\quad=\boldsymbol{VT}^-\boldsymbol{U}+\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{VT}^-\boldsymbol{U}-\boldsymbol{VT}^-\boldsymbol{UQ}^-(\boldsymbol{Q}+\boldsymbol{VT}^-\boldsymbol{U})-\\
&\quad(\boldsymbol{Q}+\boldsymbol{VT}^-\boldsymbol{U})\boldsymbol{Q}^-\boldsymbol{VT}^-\boldsymbol{U}+(\boldsymbol{Q}+\boldsymbol{VT}^-\boldsymbol{U})\boldsymbol{Q}^-(\boldsymbol{Q}+\boldsymbol{VT}^-\boldsymbol{U})\\
&\quad=\boldsymbol{VT}^-\boldsymbol{U}+\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{VT}^-\boldsymbol{U}-\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{Q}-\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{VT}^-\boldsymbol{U}-\\
&\quad\boldsymbol{QQ}^-\boldsymbol{VT}^-\boldsymbol{U}-\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{VT}^-\boldsymbol{U}+\boldsymbol{QQ}^-\boldsymbol{Q}+\boldsymbol{QQ}^-\boldsymbol{VT}^-\boldsymbol{U}+\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{Q}\\
&\quad+\boldsymbol{VT}^-\boldsymbol{UQ}^-\boldsymbol{VT}^-\boldsymbol{U}\\
&\quad=\boldsymbol{Q}+\boldsymbol{VT}^-\boldsymbol{U}=\boldsymbol{W}
\end{aligned}
$$

Hence

$$
\boldsymbol{AGA}=\begin{bmatrix}\boldsymbol{T}+(\boldsymbol{I}-\boldsymbol{TT}^-)\boldsymbol{UQ}^-\boldsymbol{V}(\boldsymbol{I}-\boldsymbol{TT}^-) & \boldsymbol{U}-(\boldsymbol{I}-\boldsymbol{TT}^-)\boldsymbol{U}(\boldsymbol{I}-\boldsymbol{QQ}^-)\\ \boldsymbol{V}-(\boldsymbol{I}-\boldsymbol{QQ}^-)\boldsymbol{V}(\boldsymbol{I}-\boldsymbol{TT}^-) & \boldsymbol{W}\end{bmatrix}
$$

which is equal to $\boldsymbol{A}$ if and only if conditions (1)-(3) hold.

(b) Verify that the conditions [$\mathcal{C}(\boldsymbol{U})\subset\mathcal{C}(\boldsymbol{T})$ and $\mathcal{R}(\boldsymbol{V})\subset\mathcal{R}(\boldsymbol{T})$] of Theorem 9.6.1 imply conditions (1)-(3).

Proof: By Lemma 9.3.5, if $\mathcal{C}(\boldsymbol{U})\subset\mathcal{C}(\boldsymbol{T})$, then $(\boldsymbol{I}-\boldsymbol{TT}^{-1})\boldsymbol{U}=0$. Also, by the same lemma, if $\mathcal{R}(\boldsymbol{V})\subset\mathcal{R}(\boldsymbol{T})$ then $\boldsymbol{V}(\boldsymbol{I}-\boldsymbol{TT}^{-1})=0$ and thus conditions (1)-(3) hold.