1. Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ represent arbitrary vectors in a linear space $\mathcal{V}$, show that

   (a) $\delta(\boldsymbol{B}, \boldsymbol{A}) = \delta(\boldsymbol{A}, \boldsymbol{B})$, that is the distance between $\boldsymbol{B}$ and $\boldsymbol{A}$ is the same as that between $\boldsymbol{A}$ and $\boldsymbol{B}$.

   Proof: By definition

$$\delta(\boldsymbol{B}, \boldsymbol{A}) = ||\boldsymbol{B} - \boldsymbol{A}|| = [(\boldsymbol{B} - \boldsymbol{A}) \bullet (\boldsymbol{B} - \boldsymbol{A})]^{1/2}$$

$$= [\boldsymbol{B} \bullet \boldsymbol{B} - 2(\boldsymbol{B} \bullet \boldsymbol{A}) + \boldsymbol{A} \bullet \boldsymbol{A}]^{1/2}$$

$$= [\boldsymbol{A} \bullet \boldsymbol{A} - 2(\boldsymbol{A} \bullet \boldsymbol{B}) + \boldsymbol{B} \bullet \boldsymbol{B}]^{1/2}$$

$$= [(\boldsymbol{A} - \boldsymbol{B}) \bullet (\boldsymbol{A} - \boldsymbol{B})]^{1/2} = ||\boldsymbol{A} - \boldsymbol{B}|| = \delta(\boldsymbol{B}, \boldsymbol{A})$$

   (b)

$$\begin{aligned} \delta(\boldsymbol{A}, \boldsymbol{B}) &> 0, \quad \text{if} \quad \boldsymbol{A} \neq \boldsymbol{B} \\ &= 0, \quad \text{if} \quad \boldsymbol{A} = \boldsymbol{B} \end{aligned}$$

   that is, the distance between any two vectors is greater than zero, unless the two vectors are identical, in which case the distance between them is zero.

   Proof: First, suppose $\boldsymbol{A} = \boldsymbol{B}$ then

$$\delta(\boldsymbol{A}, \boldsymbol{B}) = \delta(\boldsymbol{A}, \boldsymbol{A}) = ||\boldsymbol{A} - \boldsymbol{A}|| = ||0|| = 0$$

   For $\boldsymbol{A} \neq \boldsymbol{B}$, let $\boldsymbol{A'} = [A_1, A_2, \ldots, A_k]$ and $\boldsymbol{B'} = [B_1, B_2, \ldots, B_k]$ then

$$\delta(\boldsymbol{A}, \boldsymbol{B}) = ||\boldsymbol{A} - \boldsymbol{B}|| = [(\boldsymbol{A} - \boldsymbol{B}) \bullet (\boldsymbol{A} - \boldsymbol{B})]^{1/2}$$

$$= \left[\sum_{i=1}^{k} (A_i - B_i)^2\right]^{1/2} > 0$$

   since $(A_i - B_i)^2 \geq 0$ for all $i = 1, \ldots, k$ and $(A_i - B_i)^2 > 0$ for at least one i since $\boldsymbol{A} \neq \boldsymbol{B}$.

(c) $\delta(\boldsymbol{A},\boldsymbol{B}) \leq \delta(\boldsymbol{A},\boldsymbol{C}) + \delta(\boldsymbol{C},\boldsymbol{B})$, that is, the distance between $\boldsymbol{A}$ and $\boldsymbol{B}$ is less than or equal to the sum of the distances between $\boldsymbol{A}$ and $\boldsymbol{C}$ and between $\boldsymbol{C}$ and $\boldsymbol{B}$.

Proof: Note that,

$$||\boldsymbol{A}-\boldsymbol{B}||^2 = ||\boldsymbol{A}-\boldsymbol{C}+\boldsymbol{C}-\boldsymbol{B}||^2 = ||(\boldsymbol{A}-\boldsymbol{C})+(\boldsymbol{C}-\boldsymbol{B})||^2$$

$$= ||(\boldsymbol{A}-\boldsymbol{C})||^2 + 2(\boldsymbol{A}-\boldsymbol{C}) \bullet (\boldsymbol{C}-\boldsymbol{B}) + ||(\boldsymbol{C}-\boldsymbol{B})||^2$$

$$\leq ||(\boldsymbol{A}-\boldsymbol{C})||^2 + 2||\boldsymbol{A}-\boldsymbol{C}||||(\boldsymbol{C}-\boldsymbol{B})|| + ||(\boldsymbol{C}-\boldsymbol{B})||^2$$

$$= (||(\boldsymbol{A}-\boldsymbol{C})|| + ||(\boldsymbol{C}-\boldsymbol{B})||)^2$$

where the inequality follows from the Cauchy-Schwarz inequality. Thus

$$||\boldsymbol{A}-\boldsymbol{B}||^2 \leq (||(\boldsymbol{A}-\boldsymbol{C})|| + ||(\boldsymbol{C}-\boldsymbol{B})||)^2$$

and

$$||\boldsymbol{A}-\boldsymbol{B}|| \leq ||(\boldsymbol{A}-\boldsymbol{C})|| + ||(\boldsymbol{C}-\boldsymbol{B})||$$

hence

$$\delta(\boldsymbol{A},\boldsymbol{B}) \leq \delta(\boldsymbol{A},\boldsymbol{C}) + \delta(\boldsymbol{A},\boldsymbol{C})$$

(d) $\delta(\boldsymbol{A},\boldsymbol{B}) = \delta(\boldsymbol{A}+\boldsymbol{C},\boldsymbol{B}+\boldsymbol{C})$, that is distance is unaffected by a translation of 'axes'.

Proof: By definition

$$\delta(\boldsymbol{A}+\boldsymbol{C},\boldsymbol{B}+\boldsymbol{C}) = ||(\boldsymbol{A}+\boldsymbol{C})-(\boldsymbol{B}+\boldsymbol{C})|| = ||\boldsymbol{A}+\boldsymbol{C}-\boldsymbol{B}-\boldsymbol{C}|| = ||\boldsymbol{A}-\boldsymbol{B}|| = \delta(\boldsymbol{A},\boldsymbol{B})$$

2. Let $\boldsymbol{A}$ represent am $m \times n$ matrix, $\boldsymbol{C}$ an $n \times q$ matrix, and $\boldsymbol{B}$ a $q \times p$ matrix. Show that if rank($\boldsymbol{AC}$)=rank($\boldsymbol{C}$), then

$$\mathcal{R}(\boldsymbol{ACB}) = \mathcal{R}(\boldsymbol{CB}) \quad \text{and} \quad \text{rank}(\boldsymbol{ACB}) = \text{rank}(\boldsymbol{CB})$$

and that if rank($\boldsymbol{CB}$)=rank($\boldsymbol{C}$), then

$$\mathcal{C}(\boldsymbol{ACB}) = \mathcal{C}(\boldsymbol{AC}) \quad \text{and} \quad \text{rank}(\boldsymbol{ACB}) = \text{rank}(\boldsymbol{AC})$$

Proof: Note that $\mathcal{R}(\boldsymbol{ACB}) \subset \mathcal{R}(\boldsymbol{CB})$ from Corollary 4.2.3. Also, for rank($\boldsymbol{AC}$)=rank($\boldsymbol{C}$), it follows that $\mathcal{R}(\boldsymbol{AC}) = \mathcal{R}(\boldsymbol{C})$. Next, since $\mathcal{R}(\boldsymbol{C}) \subset \mathcal{R}(\boldsymbol{AC})$, it follows from Lemma 4.2.2 that there exists some matrix $\boldsymbol{L}$ such that

$$\boldsymbol{C} = \boldsymbol{LAC}$$

Thus $\mathcal{R}(\boldsymbol{CB}) = \mathcal{R}((\boldsymbol{LAC})\boldsymbol{B}) \subset \mathcal{R}(\boldsymbol{ACB})$. Therefore

$$\mathcal{R}(\boldsymbol{ACB}) = \mathcal{R}(\boldsymbol{CB})$$

and thus it also follows that rank($\boldsymbol{ACB}$)=rank($\boldsymbol{CB}$).

Again, by Corollary 4.2.3, $\mathcal{C}(\boldsymbol{ACB}) \subset \mathcal{C}(\boldsymbol{AC})$. If rank($\boldsymbol{CB}$)=rank($\boldsymbol{C}$) then $\mathcal{C}(\boldsymbol{CB}) = \mathcal{C}(\boldsymbol{C})$ and by Lemma 4.2.2 there exists some matrix $\boldsymbol{F}$ such that

$$\boldsymbol{C} = \boldsymbol{CBF}$$

Hence $\mathcal{C}(\boldsymbol{AC}) = \mathcal{C}(\boldsymbol{A}(\boldsymbol{CBF})) \subset \mathcal{C}(\boldsymbol{ACB})$. Thus $\mathcal{C}(\boldsymbol{ACB}) = \mathcal{C}(\boldsymbol{AC})$ and rank($\boldsymbol{ACB}$)=rank($\boldsymbol{AC}$).

3. Let $\boldsymbol{A}$ represent and $n \times n$ nonnull symmetric matrix, and let $\boldsymbol{B}$ represent an $n \times r$ matrix of full column rank r and $\boldsymbol{T}$ an $r \times n$ matrix of full row rank r such that $\boldsymbol{A} = \boldsymbol{BT}$. Show that the $r \times r$ matrix $\boldsymbol{TB}$ is nonsingular.

   Proof: Since $\boldsymbol{A} = \boldsymbol{A}^2$, it follows that $\boldsymbol{BT} = \boldsymbol{BTBT}$. Thus

$$r \geq rank(\boldsymbol{TB}) \geq rank(\boldsymbol{BTBT}) = rank(\boldsymbol{A}) = r$$

   Therefore $(\boldsymbol{TB})_{r\times r}$ has full rank and is nonsingular.

4. Let $\boldsymbol{I}_{k\times k}$ be the identity matrix, $\boldsymbol{J}_{k\times k}$ be a matrix of all 1's and a and b be two constants. Show that under a mild condition, the inverse of $(a-b)\boldsymbol{I} + b\boldsymbol{J}$ is also of the form $c\boldsymbol{I} + d\boldsymbol{J}$. Find c, d and the condition.

   Proof: Suppose that $c\boldsymbol{I} + d\boldsymbol{J} = [(a-b)\boldsymbol{I} + b\boldsymbol{J}]^{-1}$, then

$$(c\boldsymbol{I} + d\boldsymbol{J})\,[(a-b)\boldsymbol{I} + b\boldsymbol{J}] = \boldsymbol{I}$$

   Expanding the left side gives

$$(c\boldsymbol{I} + d\boldsymbol{J})\,[(a-b)\boldsymbol{I} + b\boldsymbol{J}] = c(a-b)\boldsymbol{I} + cb\boldsymbol{J} + d(a-b)\boldsymbol{J} + db\boldsymbol{J}^2$$

   Note that

$$\boldsymbol{J}^2 = \begin{bmatrix} \sum_{i=1}^k 1 & \sum_{i=1}^k 1 & \cdots & \sum_{i=1}^k 1 \\ \sum_{i=1}^k 1 & \sum_{i=1}^k 1 & \cdots & \sum_{i=1}^k 1 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^k 1 & \sum_{i=1}^k 1 & \cdots & \sum_{i=1}^k 1 \end{bmatrix} = \begin{bmatrix} k & k & \cdots & k \\ k & k & \cdots & k \\ \vdots & \vdots & \ddots & \vdots \\ k & k & \cdots & k \end{bmatrix} = k\boldsymbol{J}$$

   Thus

$$\begin{aligned}(c\boldsymbol{I} + d\boldsymbol{J})\,[(a-b)\boldsymbol{I} + b\boldsymbol{J}] &= c(a-b)\boldsymbol{I} + cb\boldsymbol{J} + d(a-b)\boldsymbol{J} + dbk\boldsymbol{J} \\ &= c(a-b)\boldsymbol{I} + (cb + d(a-b) + dbk)\boldsymbol{J}\end{aligned}$$

   In order for the right side to equal the identity matrix, it is necessary that

$$c(a-b) = 1 \quad \text{and} \quad (cb + d(a-b) + dbk) = 0$$

The first equality implies that $c = 1/(a-b)$, $(a-b) \neq 0$. Solving the second equality for d gives

$$cb + d(a-b) + dbk = 0$$

$$\frac{b}{a-b} + d(a-b) + dbk = 0$$

$$d[(a-b) + bk] = -\frac{b}{a-b}$$

$$d = \frac{-b}{(a-b)[a+(k-1)b]}$$

Thus $c\boldsymbol{I} + d\boldsymbol{J}$ is the inverse of $(a-b)\boldsymbol{I} + b\boldsymbol{J}$ for $c = \frac{1}{a-b}$ and $d = \frac{-b}{(a-b)[a+(k-1)b]}$, provided that $a \neq b$.