1. Let $\boldsymbol{A}$ represent a $q \times p$ matrix, $\boldsymbol{B}$ a $p \times n$ matrix and $\boldsymbol{C}$ a $m \times q$.

   (a) Show that if $\text{rank}(\boldsymbol{CAB}) = \text{rank}(\boldsymbol{C})$, then $\text{rank}(\boldsymbol{CA}) = \text{rank}(\boldsymbol{C})$.

   Proof: $\text{rank}(\boldsymbol{C}) \geq \text{rank}(\boldsymbol{CA}) \geq \text{rank}(\boldsymbol{CAB}) = \text{rank}(\boldsymbol{C})$

   (b) Show that if $\text{rank}(\boldsymbol{CAB}) = \text{rank}(\boldsymbol{B})$, then $\text{rank}(\boldsymbol{AB}) = \text{rank}(\boldsymbol{B})$.

   Proof: $\text{rank}(\boldsymbol{B}) \geq \text{rank}(\boldsymbol{AB}) \geq \text{rank}(\boldsymbol{CAB}) = \text{rank}(\boldsymbol{B})$

2. Let $\boldsymbol{A}$ represent an $n \times n$ matrix such that $\boldsymbol{A'A} = \boldsymbol{A}^2$.

   (a) Show that $\text{tr}[(\boldsymbol{A} - \boldsymbol{A}')'(\boldsymbol{A} - \boldsymbol{A}')] = 0$

   Proof:

$$
\begin{aligned}
\text{tr}[(\boldsymbol{A} - \boldsymbol{A}')'(\boldsymbol{A} - \boldsymbol{A}')] &= \text{tr}[(\boldsymbol{A}' - \boldsymbol{A})(\boldsymbol{A} - \boldsymbol{A}')] \\
&= \text{tr}[\boldsymbol{A'A} - \boldsymbol{A'A'} - \boldsymbol{AA} + \boldsymbol{AA'}] \\
&= \text{tr}[\boldsymbol{A}^2 - (\boldsymbol{A}^2)' - \boldsymbol{A}^2 + (\boldsymbol{A}^2)']
\end{aligned}
$$

   The last equality follows from $\boldsymbol{A'A} = \boldsymbol{A}^2$ and $\boldsymbol{AA'} = (\boldsymbol{A'A})' = (\boldsymbol{A}^2)'$. Therefore

$$\text{tr}[(\boldsymbol{A} - \boldsymbol{A}')'(\boldsymbol{A} - \boldsymbol{A}')] = \text{tr}(\boldsymbol{0}) = 0$$

   (b) Show that $\boldsymbol{A}$ is symmetric.

   Proof: For any $m \times n$ matrix $\boldsymbol{A}$, $\boldsymbol{A} = \boldsymbol{0}$ iff $\text{tr}(\boldsymbol{A'A}) = 0$. Since

$$\text{tr}[(\boldsymbol{A} - \boldsymbol{A}')'(\boldsymbol{A} - \boldsymbol{A}')] = 0$$

   it follows that $\boldsymbol{A} - \boldsymbol{A}' = 0$ and therefore $\boldsymbol{A} = \boldsymbol{A}'$.

3. Suppose matrix $\boldsymbol{A}_{n \times n}$ satisfied $\boldsymbol{A}^2 = \boldsymbol{A}$.

(i) Show that rank($\boldsymbol{A}$) = tr($\boldsymbol{A}$).

Proof: **Theorem 4.4.9** states that if $\boldsymbol{A}$ is an $m \times n$ non-null matrix of rank r, then there exists an $m \times m$ nonsingular matrix $\boldsymbol{B}$ and an $n \times n$ nonsingular matrix $\boldsymbol{K}$ such that

$$\boldsymbol{A} = \boldsymbol{B} \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \boldsymbol{K}$$

Thus

$$\boldsymbol{A} = \boldsymbol{B} \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \boldsymbol{K} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{K}_1 \\ \boldsymbol{K}_2 \end{bmatrix} = \boldsymbol{B}_1 \boldsymbol{K}_1$$

From **Theorem 4.4.8**, $\boldsymbol{B}_1$ is $n \times r$ and $\boldsymbol{K}_1$ is $r \times n$ and both matrices have rank r. Now since $\boldsymbol{A} = \boldsymbol{A}^2$,

$$\boldsymbol{B}_1 \boldsymbol{K}_1 = \boldsymbol{B}_1 \boldsymbol{K}_1 \boldsymbol{B}_1 \boldsymbol{K}_1$$

The product $\boldsymbol{K}_1 \boldsymbol{B}_1$, which is $r \times r$, must have full rank since $\boldsymbol{B}_1 \boldsymbol{K}_1 = \boldsymbol{A}$ has rank r. From the above equation, it follows that

$$\boldsymbol{K}_1 \boldsymbol{B}_1 \boldsymbol{K}_1 = \boldsymbol{K}_1$$

Thus

$$\boldsymbol{K}_1 \boldsymbol{B}_1 = \boldsymbol{I}_r$$

Using the relation tr($\boldsymbol{AB}$) = tr($\boldsymbol{BA}$)

$$\mathrm{tr}(\boldsymbol{A}) = \mathrm{tr}(\boldsymbol{B}_1 \boldsymbol{K}_1) = \mathrm{tr}(\boldsymbol{K}_1 \boldsymbol{B}_1) = \mathrm{tr}(\boldsymbol{I}_r) = r$$

Therefore rank(A) = tr(A) = r

(ii) Let $\boldsymbol{X}$ be an $n \times k$ matrix where $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ exists. Show that the rank of the matrix $\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$ is $n - k$.

Proof: Note that

$$(\boldsymbol{I}_n - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}')(\boldsymbol{I}_n - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}') = (\boldsymbol{I}_n - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}')$$

By(i), we have rank($\boldsymbol{I}_n - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$)=tr($\boldsymbol{I}_n - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$) $= n - k$