1. Suppose by a sequence of elementary row operations E_1, E_2,...,E_n, we can transform a square matrix A into a matrix B which has a row of all zeros, i.e.,

$$E_n \dots E_2 E_1 A = B$$

where B has a 0 row. Show that there is no way we can find another sequence of elementary row operations to transfer A into the identity matrix.

Hint: Show first that elementary row operations cannot change the independence-dependence status of row vectors in the matrix it operates on.

2. For what values of the scalar k are the three row vectors $(k, 1, 0)$, $(1, k, 1)$, and $(0, 1, k)$ linearly dependent, and for what values are they linearly independent?

Hint: Let $a_1' = (k, 1, 0)$, $a_2' = (1, k, 1)$, $a_3' = (0, 1, k)$ and define the matrix A as

$$A = \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix} = \begin{bmatrix} k & 1 & 0 \\ 1 & k & 1 \\ 0 & 1 & k \end{bmatrix}$$

By ex1, elementary row operations cannot change the independence-dependence status of row vectors in the matrix it operates on. We do elementary row operations on A and obtain

$$\begin{bmatrix} 1 & k & 1 \\ 0 & 1 & k \\ 0 & 0 & k(k^2-2) \end{bmatrix}$$

3. Let A, B and C represent three linearly independent vectors. Determine whether or not the pairwise sums $A+B$, $A+C$ and $B+C$ are linearly independent. Thus the vectors $(k, 1, 0)$, $(1, k, 1)$ and $(0, 1, k)$ will be linearly dependent for the values of k that satisfy

$$k(k^2 - 2) = 0$$

which will result in the third row being all zeros. This occurs when $k = 0, \pm\sqrt{2}$. For all other real numbers the vectors will be linearly independent.

Solution:

$$x_1(A+B) + x_2(A+C) + x_3(B+C) = 0$$
$$\Rightarrow (x_1+x_2)A + (x_1+x_3)B + (x_2+x_3)C = 0$$
$$\Rightarrow x_1+x_2 = x_1+x_3 = x_2+x_3 = 0$$

, because A,B and C are linearly independent. Thus, $x_1 = x_2 = x_3 = 0$ and $A+B$, $A+C$ and $B+C$ are linearly independent.