1. Let A be a square matrix. Suppose by a sequence of elementary row operations, E_1, E_2,..., E_k, we have $E_k E_{k-1} \dots E_1 A = I$. Show that there exists a square matrix B such that $AB = BA = I$.

   Proof:
   Let $B = E_k E_{k-1} \dots E_1$. Since E_1, E_2,..., E_k are elementary row operation, there exits a square matrix C s.t. $CB = I$. Thus $BA = CB = I$. Note that $C = CI = C(BA) = (CB)A = IA = A$. Hence $AB = BA = I$.

2. We say that a square $n \times n$ matrix A has inverses on both sides if there exists two $n \times n$ matrices L and R such that $LA = AR = I$ (L and R stand for left and right inverses respectively). Suppose that A has inverses on both sides show that

   (i) $L = R$. We can denote both them as A^{-1}.

   (ii) Show that if A is symmetric, then A^{-1} is also symmetric.

   Proof:
   (i) $L = LI = L(AR) = (LA)R = IR = R$.
   (ii) $AA^{-1} = A^{-1}A = I \Rightarrow (A^{-1})'A = A(A^{-1})' = I$. By (i), $(A^{-1})' = A^{-1}$.

3. For the linear model $Y = X\alpha + \boldsymbol{\epsilon}$, Suppose we estimate α as

   $$\hat{\alpha} = (X'X)^{-1}X'Y$$

   and $(X'X^{-1}$ exists. The error estimate is defined by

   $$\hat{\epsilon} = Y - X\hat{\alpha}$$

   use the previous problem to show that

   $$\hat{\epsilon}'\hat{\epsilon} = Y'(I - X(X'X)^{-1}X')Y$$

   Proof:
   By definition

   $$\hat{\epsilon}'\hat{\epsilon} = (Y - X\hat{\alpha})'(Y - X\hat{\alpha}) = (Y' - \hat{\alpha}'X')(Y - X\hat{\alpha})$$

   Substituting $\hat{\alpha} = (X'X)^{-1}X'Y$ gives

   $$\begin{aligned}\hat{\epsilon}'\hat{\epsilon} &= [Y' - ((X'X)^{-1}X'Y)'X'][Y - X(X'X)^{-1}X'Y] \\ &= [Y' - Y'X((X'X)^{-1})'X'][Y - X(X'X)^{-1}X'Y] \\ &= Y'[I - X((X'X)^{-1})'X'][I - X(X'X)^{-1}X']Y\end{aligned}$$

Note that $X'X$ is a symmetric matrix since

$$X'X' = X'(X')' = X'X$$

Thus by the previous problem, $((X'X)^{-1})'$ is symmetric. Hence

$$\hat{\epsilon}'\hat{\epsilon} = Y'[I - X(X'X)^{-1}X'][I - X(X'X)^{-1}X']Y$$

Multiplying the matrices within the brackets gives

$$\begin{aligned}\hat{\epsilon}'\hat{\epsilon} &= Y'[I - X(X'X)^{-1}X' - X(X'X)^{-1}X' + X(X'X)^{-1}X'X(X'X)^{-1}X']Y \\ &= Y'[I - 2X(X'X)^{-1}X' + X(X'X)^{-1}IX']Y\end{aligned}$$

Which simplifies to

$$\hat{\epsilon}'\hat{\epsilon} = Y'[I - X(X'X)^{-1}X']Y$$

5. A fair coin is tossed until $m = 5$ consecutive heads are obtained. What is the probability that 5 consecutive heads will be reached in no more than 10 tosses. What is the probability that 5 consecutive heads will be reached in exactly 10 tosses. What is the probability that 5 consecutive heads will appear at least once in 100 tosses.

Solution:
We construct the Markov chain probability transition matrix A with 6 states: 0,1,2,3,4,5+, with $A(i,j) = 0$, except $A(i,j) = 1/2$ for $j = 0$ and $i = 0,1,2,3,4$, and $A(i,i+1) = 1/2$ for $i = 0,1,2,3,4$. Also $A(5+,5+) = 1$.

The initial probability distribution at each state is $x(0) = [1,0,0,0,0,0]$ and $x(n) = x(0)A^n$. The answer for stop with no more than 10 tosses in 0.109375, and stop at exact 10 tosses is 0.015625.