1. Show that for any $n \times m$ matrix, $A = \{a_{ij}\}$, $n \times q$ matrix $B = \{b_{jk}\}$, and $q \times r$ matrix $C = \{c_{ks}\}$, $A(BC) = (AB)C$.

   Solution: Note that

$$(BC)(j,s) = \sum_{p=1}^{q} b_{jp}c_{ps} \qquad \text{and} \qquad (AB)(i,k) = \sum_{p=1}^{n} a_{ip}b_{pk}$$

   Thus

$$(A(BC))(i,s) \;=\; \sum_{t=1}^{n} a_{it}x_{ts} \;=\; \sum_{t=1}^{n} a_{it} \sum_{l=1}^{q} b_{tl}c_{ls} \;=\; \sum_{t=1}^{n}\sum_{l=1}^{q} a_{it}b_{tl}c_{ls}$$

   Changing the order of summation gives

$$\sum_{t=1}^{n}\sum_{l=1}^{q} a_{it}b_{tl}c_{ls} \;=\; \sum_{l=1}^{q}\sum_{t=1}^{n} a_{it}b_{tl}c_{ls} \;=\; \sum_{l=1}^{q} c_{ls} \sum_{t=1}^{n} a_{it}b_{tl}$$

   From (),

$$\sum_{l=1}^{q} c_{ls} \sum_{t=1}^{n} a_{it}b_{tl} \;=\; \sum_{l=1}^{q} y_{il}c_{ls} \;=\; ((AB)C)(i,s)$$

   Hence

$$(A(BC))(i,s) \;=\; ((AB)C)(i,s) \quad \text{for} \quad i \in \{1,\ldots,m\},\; s \in \{1,\ldots,r\}$$

   and therefore $A(BC) = (AB)C$.

2. Let A and B represent $m \times n$ matrices. Show that

$$(A+B)(A-B) = A^2 - B^2$$

   if and only if A and B commute.

   Solution: $(A+B)(A-B) = A^2 - AB + BA - B^2$.
   Thus $(A+B)(A-B) = A^2 - B^2 \Leftrightarrow BA - AB = 0 \Leftrightarrow BA = AB$.

3. (a) Show that the product of AB of two $n \times n$ symmetric matrices A and B is itself symmetric if and only if A and B commute.

   Solution: Note that $(AB)' = B'A' = BA$. Thus $(AB)' = AB \Leftrightarrow AB = BA$

(b) Give an example of two symmetric matrices whose product is not symmetric.

Solution: Define the matrices A and B as

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Then,

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

which is not a symmetric matrix.

4. Let (x, y) be a point in the two-dimensional space. When the axes are rotated an angle θ counter-clockwise, this point has a new coordinate (x', y') with the new axes satisfying

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(a) Derive this formula.

Solution: Let (x, y) be a point in the two-dimensional space. In polar coordinates

$$x = r\cos\phi \quad \text{and} \quad y = r\sin\phi \tag{1}$$

where $r = \sqrt{x^2 + y^2}$ and ϕ is the angle between the x-axis and the line extending from the origin to the point (x, y). The the new coordinates, (x', y'), of the rotated point will then be

$$x' = r\cos(\phi + \theta) \quad \text{and} \quad y' = r\sin(\phi + \theta)$$

Using the following trigonometric identities

$$\cos(\phi + \theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$$
$$\sin(\phi + \theta) = \cos\phi\sin\theta + \sin\phi\cos\theta$$

we can express the values of x' and y' as

$$x' = r\cos\phi\cos\theta - r\sin\phi\sin\theta$$
$$y' = r\cos\phi\sin\theta + r\sin\phi\cos\theta$$

From (1)

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

Putting the system of equations in matrix form gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(b) Show that if the axes are rotated twice with angles θ_1 and θ_2, then the coordinate transformation is the same by matrix multiplication, or by direct application of the formula with angle $\theta_1 + \theta_2$.

Solution: Using matrix multiplication, the coordinates of the transformed point, (x', y') will be

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Multiplying the matrices on the right side gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & -\cos\theta_2\sin\theta_1 - \sin\theta_2\cos\theta_1 \\ \sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1 & \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Using the trigonometric identities from (a), the transformation matrix can be simplified giving

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

which is the equation for a single rotation using the angle $\theta_1 + \theta_2$.

(c) What is an intuitive inverse for the rotation matrix? Show that it works by multiplication.

Solution: The rotation matrix rotates a point in the two-dimensional plane counter-clockwise by angle θ. An intuitive inverse would be the rotation matrix for angle $-\theta$. Thus the inverse should be

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Check using multiplication:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\cos\theta\sin\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The product of the two matrices is the identity matrix thus

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is the inverse of the rotation matrix.