1. Let $\boldsymbol{A}$ be a square matrix. Suppose by a sequence of elementary row operations $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, ..., $\boldsymbol{E}_k$, we have $\boldsymbol{E}_k\boldsymbol{E}_{k-1}...\boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{I}$. Show that there exists a sqauare martix $\boldsymbol{B}$ such that $\boldsymbol{BA} = \boldsymbol{AB} = \boldsymbol{I}$.

2. We say that square $n \times n$ matrix $\boldsymbol{A}$ has inverses on both sides if there exists two $n \times n$ matrices $\boldsymbol{R}$ and $\boldsymbol{L}$ such that $\boldsymbol{LA} = \boldsymbol{AR} = \boldsymbol{I}$ (L and R stand for right and left respectively). Suppose $\boldsymbol{A}$ has inverses on both sides show that
(i) $\boldsymbol{L} = \boldsymbol{R}$. We can then denote both them as $\boldsymbol{A}^{-1}$. (ii) Show that if $\boldsymbol{A}$ is symmetric, then $\boldsymbol{A}^{-1}$ is also symmetric.

3. In linear regression setup, the relation between data x_{ij} and observations y_i, $i = 1, 2, ..., n$. can be written as linear equations

$$\begin{aligned} y_1 &= \alpha_1 x_{11} + \alpha_2 x_{12} + \cdots + \alpha_k x_{1k} + \epsilon_1 \\ y_2 &= \alpha_1 x_{21} + \alpha_2 x_{22} + \cdots + \alpha_k x_{2k} + \epsilon_2 \\ &\vdots \\ y_n &= \alpha_1 x_{n1} + \alpha_2 x_{n2} + \cdots + \alpha_k x_{nk} + \epsilon_3 \end{aligned}$$

or in matrix form,

$$\boldsymbol{Y} = \boldsymbol{X\alpha} + \boldsymbol{\epsilon}.$$

Suppose $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ exists. Then a reasonable estimate for $\widehat{\boldsymbol{\alpha}}$ is

$$\widehat{\boldsymbol{\alpha}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}.$$

The error estimate (*residuals*) is defined by

$$\widehat{\boldsymbol{\epsilon}} = \boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\alpha}}.$$

Use the previous problem to show that

$$\widehat{\boldsymbol{\epsilon}}'\widehat{\boldsymbol{\epsilon}} = \boldsymbol{Y}'(\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}')\boldsymbol{Y},$$

which is called the error sum of squares.

The next two problem requires to use R. Please hand in the computer inputs and outputs.

4. Write a subroutine with input of two matrices A, B and output their product C=AB. The subroutine should have a warning message if the dimensions of A and B do not allow multiplication. So the input is A and B and output is either a warning message or C=AB. Test this subroutine by the matrices in Problem 6 of the previous exercies (Exercise 1).

5. I will continue toss a fair coin until I see $m = 5$ consecutive heads. In other words, I will stop tossing if I see 5 heads in a roll. What is the probability that I will stop with no more than $n = 10$ tosses? What is the probability to stop at the exact 10 tosses? What is the probability that I will see five consecutive heads at least once in 100 tosses? Write your program as a subroutine so it can solve for any m and n. (Hint: Use a Markov chain matrix with 6 states of the consecutive heads you already have. i.e, $\{S\} = 0, 1, 2, 3, 4$ and 5^+.)