STA4322 Exam1 (closed book, 9:30 - 10:45, 7/10/2009) Name______________________

1. Let $Y \sim N(5,4)$. Find the probability $P(|Y-3|<1)$. (10%)

   $P(|Y-3|<1) = P(-1<Y-3<1) = p(2<Y<4) = P(-1.5<Z<-0.5) = 0.2417$

2. (i) Use standard normal random variable to define a chi-square random variable with k degrees of freedom (10%)

   Let $Z_1, Z_2, ..., Z_k$ be k independent standard normal r.v., then

$$W = \sum_{i=1}^{k} Z_i \sim \chi^2_k$$

.

(ii) Use chi-square random variable to define an F distribution with k_1 and k_2 degrees of freedom. (10%)

Let $W_1 \sim \chi^2_{k_1}$ and $W_2 \sim \chi^2_{k_2}$ and they are independent, then

$$\frac{W_1/k_1}{W_2/k_2} \sim F^{k_1}_{k_2}.$$

(iii) Define the sample variance from data Y_1, Y_2, ..., Y_n. (5%)

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}[Y_1 - \bar{Y}]^2.$$

3. A fair coin will be tossed 100 times. Find the the probability that there will be less than 40 heads. Continuity correction is required. (10%)

$$Y \sim Bi(100, 0.5) \sim N(50, 25)$$
$$P(Y<40) = P(Y \le 39.5) = P(Z \le -2.1) = 0.0179.$$

4. The reading of a micrometer is a normal random variable with mean of the true length of the object and standard deviation 0.5mm. We intend to measure an object n times and to use the average as the true length. How large should n be so that we are 99% sure the measurement error is less than 0.1mm? (15%)

   Given: $Y \sim N(\mu, 0.5^2)$. To find n such that $P(|\bar{Y}-\mu|<0.1) = 0.99$.

   This is equivalent to $P(|Z| < \frac{0.1}{0.5/\sqrt{n}}) = 0.99$ or $\frac{0.1}{0.5/\sqrt{n}} = 2.575$ which implies $n = 166$.

5. Twenty five heat lamps are connected in a green house so that one lamp fails, another takes over immediately. Also, only one lamp is turned on at any time. Each lamp has mean life 41 hours with standard deviation of 4 hours.
(i) What is the probability that the twenty five lamps will last more than 1,000 hours. (10%)

Let $T =$ the total illumination time $= Y_1 + ... + Y_n$, with $n = 25$.

By CLT, $Y \sim N(25 \times 41, 24 \times 4^2) = N(1025, 20^2)$.

$$P(T > 1000) = P(Z > \frac{1000 - 1025}{20}) = P(Z > -1.25) = 0.89.$$

(ii) If we want a 99% probability for the lamps to last more than 1,000 hours. How many lamps do we need to store in the green house? (10%)

We wish to find n such that $P(T > 1000) = 0.99$. This is equivalent to

$$P(Z > \frac{1000 - 41n}{4\sqrt{n}}) = 0.99, \Rightarrow \frac{1000 - 41n}{4\sqrt{n}} = -2.33.$$

This becomes the equation $1000 - 41n = -9.32\sqrt{n}$. Solving this quadratic equation, we have

$$\sqrt{n} = \frac{9.32 \pm \sqrt{9.32^2 + 4 \times 1000 \times 41}}{2 \times 41} = 5.05.$$

Thus we need $n = 5.05^2 = 25.5$ or 26.

6. The natural cure rate of a disease is 50%. A drug is invented and to be tested in a clinical trial with n patients. The clinical trial should satisfied two requirements. (1) The risk (probability) of claiming an ineffective drug as effective should be less than 0.01, and (2) If the drug has cure rate is 0.6 or higher, we wish to have risk (probability) smaller than 0.05 to miss it. Determine the sample size n for this clinical trial. (20%)
Let the number of cured be Y and we claim the drug is effective if $Y \geq c$. Let p be the true cure rate. Then (1) is equivalent to

$$P(Y \geq c|p = 0.5) = 0.01 \Rightarrow P(Z \geq \frac{c - 0.5n}{0.5\sqrt{n}}) = 0.01 \Rightarrow \frac{c - 0.5n}{0.5\sqrt{n}} = 2.33$$

This is the equation

$$c - 0.5n = 1.165\sqrt{n} \qquad (1)$$

Requirement (2) is equivalent to

$$P(Y \geq c|p = 0.6) = 0.95 \Rightarrow P(Z \geq \frac{c - 0.6n}{\sqrt{0.24n}}) = 0.95 \Rightarrow \frac{c - 0.6n}{\sqrt{0.24n}} = -1.645$$

This is the equation

$$c - 0.6n = -0.806\sqrt{n} \tag{2}$$

Canceling c from (1) and (2), we have

$$0.1n = 1.971\sqrt{n} \Rightarrow n = 389.$$