Homework 4

9.3 (a) $E(\hat{\theta}_1) = E(\bar{Y}) - \frac{1}{2} = \theta + 1/2 - 1/2 = \theta$. We can find the density function of $\hat{\theta}_2 = Y_{(n)}$: $g_n(y) = n(y-\theta)^{n-1}, \theta \le y \le \theta + 1$. From this, it is easily shown that $E(\hat{\theta}_2) = E(Y_{(n)}) - n/(n+1) = \theta$.

(b) $V(\hat{\theta}_1) = V(\bar{Y}) = \sigma^2/n = 1/(12n)$. With the density in part a, $V(\hat{\theta}_2) = V(Y_{(n)}) = \frac{n}{(n+2)(n+1)^2}$. Thus , $eff(\hat{\theta}_1, \hat{\theta}_2) = \frac{12n^2}{(n+2)(n+1)^2}$.

9.7 The estimator $\hat{\theta}_1$ is unbiased so $MSE(\hat{\theta}_1) = V(\hat{\theta}_1) = \theta^2$. Also, $\hat{\theta}_2 = \bar{Y}$ is unbiased for θ and $V(\hat{\theta}_2) = \sigma^2/n = \theta^2/n$. Thus, we have that $eff(\hat{\theta}_1, \hat{\theta}_2) = 1/n$

9.21 Note that this is a generalization of Ex. 9.5. The estimator $\hat{\sigma}^2$ can be written as

$$\hat{\sigma}^2 = \frac{1}{k}[\frac{(Y_2 - Y_1)^2}{2} + \frac{(Y_4 - Y_3)^2}{2} + \frac{(Y_6 - Y_5)^2}{2} + \ldots + \frac{(Y_n - Y_{n-1})^2}{2}]$$

. There are k independent terms in the sum, each with mean σ^2 and variance $2\sigma^4$.

(a) From the above, $E(\hat{\sigma}^2) = (k\sigma^2)/k = \sigma^2$. So $\hat{\sigma}^2$ is an unbiased estimator.

(b) Similarly, $V(\hat{\sigma}^2) = k(2\sigma^4)/k^2 = 2\sigma^4/k$. Since $k = n/2$, $V(\hat{\sigma}^2)$ goes to 0 with n and $\hat{\sigma}^2$ is a consistent estimator.

9.65 (a) $E(T) = P(T = 1) = P(Y_1 = 1, Y_2 = 0) = P(Y_1 = 1)P(Y_2 = 0) = p(1-p)$.

(b) $P(T = 1|W = w) = \frac{P(Y_1=1, Y_2=0, W=w)}{P(W=w)} = \frac{P(Y_1=1, Y_2=0, \sum_{i=3}^{n} Y_i = w-1}{P(W=w)} = \frac{P(Y_1=1)P(Y_2=0)P(\sum_{i=3}^{n} Y_i = w-1}{P(W=w)} = \frac{w(n-w)}{n(n-1)}$

(c) $E(T|W) = P(T = 1|W) = \frac{n}{n-1}\frac{W}{n}(1 - \frac{W}{n})$. Since T is unbiased by part(a) above and W is sufficient for p and so also for $p(1-p)$, $n\bar{Y}(1-\bar{Y})/(n-1)$ is the MVUE for $p(1-p)$.

9.82 The likelihood function is $L(\theta) = \theta^{-n} r^n (\Pi_{i=1}^{n} y_i)^{r-1} exp(-\sum_{i=1}^{n} y_i^r/\theta)$.

(a) By Thm 9.4, a sufficient statistic for θ is $\sum_{i=1}^{n} Y_i^r$.

(b) The log-likelihood is

$$\ln L(\theta) = -n \ln \theta + n \ln r + (r-1) \ln(\Pi_{i=1}^{n} y_i) - \sum_{i=1}^{n} y_i^r/\theta$$

. By taking a derivative w.r.t θ and equating to 0, we find $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} Y_i^r$.

(c) Note that $\hat{\theta}$ is function of the sufficient statistic. Since it is easily shown that $E(Y^r) = \theta$, $\hat{\theta}$ is then unbiased and the MVUE for θ.

9.88 The likelihood function is $L(\theta) = (\theta + 1)^n (\Pi_{i=1}^{n} y_i)^\theta$. The MLE is $\hat{\theta} = -n/\sum_{i=1}^{n} \ln Y_i - 1$. This is a different estimator that the MOM estimator from Ex.9.69, however note that the MLE is a function of the sufficient statistic.