Large-Sample α-Level Hypothesis Tests

$H_0 : \theta = \theta_0$.

$$H_a : \begin{cases} \theta > \theta_0 & \text{(upper-tail alternative).} \\ \theta < \theta_0 & \text{(lower-tail alternative).} \\ \theta \neq \theta_0 & \text{(two-tailed alternative).} \end{cases}$$

Test statistic: $Z = \dfrac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}}$.

$$\text{Rejection region: } \begin{cases} \{z > z_\alpha\} & \text{(upper-tail RR).} \\ \{z < -z_\alpha\} & \text{(lower-tail RR).} \\ \{|z| > z_{\alpha/2}\} & \text{(two-tailed RR).} \end{cases}$$

A Small-Sample Test for μ

Assumptions: $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from a normal distribution with $E(Y_i) = \mu$.

$H_0 : \mu = \mu_0$.

$$H_a : \begin{cases} \mu > \mu_0 & \text{(upper-tail alternative).} \\ \mu < \mu_0 & \text{(lower-tail alternative).} \\ \mu \neq \mu_0 & \text{(two-tailed alternative).} \end{cases}$$

Test statistic: $T = \dfrac{\overline{Y} - \mu_0}{S/\sqrt{n}}$.

$$\text{Rejection region: } \begin{cases} t > t_\alpha & \text{(upper-tail RR).} \\ t < -t_\alpha & \text{(lower-tail RR).} \\ |t| > t_{\alpha/2} & \text{(two-tailed RR).} \end{cases}$$

(See Table 5, Appendix 3, for values of t_α, with $\nu = n - 1$ df.)

Small-Sample Tests for Comparing Two Population Means

Assumptions: Independent samples from normal distributions with $\sigma_1^2 = \sigma_2^2$.

$H_0: \mu_1 - \mu_2 = D_0$.

$$H_a: \begin{cases} \mu_1 - \mu_2 > D_0 & \text{(upper-tail alternative).} \\ \mu_1 - \mu_2 < D_0 & \text{(lower-tail alternative).} \\ \mu_1 - \mu_2 \neq D_0 & \text{(two-tailed alternative).} \end{cases}$$

Test statistic: $T = \dfrac{\overline{Y}_1 - \overline{Y}_2 - D_0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$, where $S_p = \sqrt{\dfrac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}}$.

$$\text{Rejection region: } \begin{cases} t > t_\alpha & \text{(upper-tail RR).} \\ t < -t_\alpha & \text{(lower-tail RR).} \\ |t| > t_{\alpha/2} & \text{(two-tailed RR).} \end{cases}$$

Here, $P(T > t_\alpha) = \alpha$ and degrees of freedom $\nu = n_1 + n_2 - 2$. (See Table 5, Appendix 3.)

Test of Hypotheses Concerning a Population Variance

Assumptions: $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from a normal distribution with

$$E(Y_i) = \mu \quad \text{and} \quad V(Y_i) = \sigma^2.$$

$H_0: \sigma^2 = \sigma_0^2$

$$H_a: \begin{cases} \sigma^2 > \sigma_0^2 & \text{(upper-tail alternative).} \\ \sigma^2 < \sigma_0^2 & \text{(lower-tail alternative).} \\ \sigma^2 \neq \sigma_0^2 & \text{(two-tailed alternative).} \end{cases}$$

Test statistic: $\chi^2 = \dfrac{(n-1)S^2}{\sigma_0^2}$.

$$\text{Rejection region: } \begin{cases} \chi^2 > \chi_\alpha^2 & \text{(upper-tail RR).} \\ \chi^2 < \chi_{1-\alpha}^2 & \text{(lower-tail RR).} \\ \chi^2 > \chi_{\alpha/2}^2 \text{ or } \chi^2 < \chi_{1-\alpha/2}^2 & \text{(two-tailed RR).} \end{cases}$$

Notice that χ_α^2 is chosen so that, for $\nu = n - 1$ df, $P(\chi^2 > \chi_\alpha^2) = \alpha$. (See Table 6, Appendix 3.)

Test of the Hypothesis $\sigma_1^2 = \sigma_2^2$

Assumptions: Independent samples from normal populations.
$H_0 : \sigma_1^2 = \sigma_2^2$.
$H_a : \sigma_1^2 > \sigma_2^2$.
Test statistic: $F = \dfrac{S_1^2}{S_2^2}$.
Rejection region: $F > F_\alpha$, where F_α is chosen so that $P(F > F_\alpha) = \alpha$ when F has $\nu_1 = n_1 - 1$ numerator degrees of freedom and $\nu_2 = n_2 - 1$ denominator degrees of freedom. (See Table 7, Appendix 3.)

If we wish to test $H_0 : \sigma_1^2 = \sigma_2^2$ versus $H_a : \sigma_1^2 \neq \sigma_2^2$ with type I error probability α, we can employ $F = S_1^2/S_2^2$ as a test statistic and reject H_0 in favor of H_a if the calculated F-value is in either the upper or the lower $\alpha/2$ tail of the F distribution. The upper-tail critical values can be determined directly from Table 7, Appendix 3; but how do we determine the lower-tail critical values?

Notice that $F = S_1^2/S_2^2$ and $F^{-1} = S_2^2/S_1^2$ both have F distributions, but the numerator and denominator degrees of freedom are interchanged (the process of inversion switches the roles of numerator and denominator). Let F_b^a denote a random variable with an F distribution with a and b numerator and denominator degrees of freedom, respectively, and let $F_{b,\alpha/2}^a$ be such that

$$P\left(F_b^a > F_{b,\alpha/2}^a\right) = \alpha/2.$$

Then

$$P\left[\left(F_b^a\right)^{-1} < \left(F_{b,\alpha/2}^a\right)^{-1}\right] = \alpha/2$$

and, therefore,

$$P\left[F_a^b < \left(F_{b,\alpha/2}^a\right)^{-1}\right] = \alpha/2.$$

That is, the value that cuts off a lower-tail area of $\alpha/2$ for an F_a^b distribution can be found by inverting $F_{b,\alpha/2}^a$. Thus, if we use $F = S_1^2/S_2^2$ as a test statistic for testing $H_0 : \sigma_1^2 = \sigma_2^2$ versus $H_a : \sigma_1^2 \neq \sigma_2^2$, the appropriate rejection region is

two tailed →

$$\text{RR}: \left\{F > F_{n_2-1,\alpha/2}^{n_1-1} \quad \text{or} \quad F < \left(F_{n_1-1,\alpha/2}^{n_2-1}\right)^{-1}\right\}.$$

An equivalent test (see Exercise 10.81) is obtained as follows. Let n_L and n_S denote the sample sizes associated with the larger and smaller sample variances, respectively. Place the larger sample variance in the numerator and the smaller sample variance in the denominator of the F statistic, and reject $H_0 : \sigma_1^2 = \sigma_2^2$ in favor of $H_a : \sigma_1^2 \neq \sigma_2^2$ if $F > F_{\alpha/2}$, where $F_{\alpha/2}$ is determined for $\nu_1 = n_L - 1$ and $\nu_2 = n_S - 1$ numerator and denominator degrees of freedom, respectively.

EXAMPLE 10.21 An experiment to explore the pain thresholds to electrical shocks for males and females resulted in the data summary given in Table 10.4. Do the data provide sufficient evidence to indicate a significant difference in the variability of pain thresholds for men and women? Use $\alpha = .10$. What can be said about the p-value?