STA 6329 - Exam 1 PRACTICE Problems

1. Bagel shop makes 3 types of Bagels (B_1, B_2, B_3)

Each Bagel type makes use of the same 4 ingredients (different amounts)

$(I_1, I_2, I_3, I_4)_{B_1}$

$\ldots (I_1, I_2, I_3, I_4)_{B_3}$

Let $\underset{3\times 4}{A} \equiv$ matrix of ingredient amounts (cols) by bagel type (rows) per unit bagel: $A = \{a_{ij}\}$

Unit prices per ingredients are: (P_1, P_2, P_3, P_4)

Let $\underset{4\times 1}{\underset{\sim}{b}} \equiv$ vector of ingredient unit prices. $\underset{\sim}{b} = \{b_j\}$

The number of bagels of each type to be produced for a catering job are: (N_1, N_2, N_3)

Let $\underset{3\times 1}{\underset{\sim}{c}} \equiv$ vector of number of bagels to be produced $\underset{\sim}{c} = \{c_i\}$

Set this up as a matrix problem to obtain:

1) the total cost of the production run,
2) What to charge if you wish to make a 10% Profit.

Be careful of order of multiplication by conformity

a) Symbolically in terms of $\{a_{ij}\}, \{b_j\}, \{c_i\}$

b) For the matrices: $A = \begin{bmatrix} .4 & .2 & .3 & .1 \\ .2 & .2 & .3 & .3 \\ .1 & .2 & .3 & .4 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} 1.20 \\ 2.10 \\ 0.50 \\ 1.40 \end{bmatrix}$ $\underset{\sim}{c} = \begin{bmatrix} 100 \\ 200 \\ 150 \end{bmatrix}$

2. Consider the partitioned matrices

$$\underset{m \times n}{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \qquad \begin{array}{l} A_{ij} \equiv m_i \times n_j \\ m_1 + m_2 = m \\ n_1 + n_2 + n_3 = n \end{array} \qquad \underset{n \times p}{B} = \begin{bmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \\ B_{13} & B_{23} & B_{33} \end{bmatrix}$$

$$\begin{array}{l} B_{ij} \equiv n_i \times p_j \\ n_1 + n_2 + n_3 = n \\ p_1 + p_2 + p_3 = p \end{array}$$

a) What will be the dimension of AB?

b) How many "blocks" of elements will there be in AB?

c) Give AB in terms of the $\{A_{ij}\}$, $\{B_{ij}\}$

3. Let $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{e_1} & \underset{\sim}{e_2} \end{bmatrix}$ $\qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{u_1}' \\ \underset{\sim}{u_2}' \\ \underset{\sim}{u_3}' \end{bmatrix}$

Let $\underset{m \times n}{U_{ij}} = \underset{\sim}{e_i} \underset{\sim}{u_j}'$ $\quad i = 1, 2;\ j = 1, 2, 3$

Write out $\underset{2 \times 3}{A} = \{a_{ij}\}$ as $\sum_{i=1}^{2} \sum_{j=1}^{3} a_{ij} U_{ij}$

4. Suppose $\sum_{i=1}^{k} x_i A_i = 0$ where $A_i \equiv m \times n$, $x_i \equiv$ scalar

If $x_i \neq 0$, write A_i as a linear function of $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_k$

5. For what values of k are $\begin{bmatrix} k \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ k \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ k \end{bmatrix}$ linearly dependent? linearly independent?

6. $A, B, C \equiv$ lin. indep. $m \times n$ matrices.
Are $D_1 = A + B$, $D_2 = A + C$, $D_3 = B + C$ linearly independent?

7. Let $\mathcal{V} \equiv$ set of all 2×2 matrices such that all elements are ~~[illegible]~~ even numbers. Is $\mathcal{V}$ a linear space?

(8) For the matrix $A = \begin{bmatrix} 1 & 0 & 5 \\ -7 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix}$

a) give the "form" of all ~~matrices~~ 3x1 vectors in its column space

b) Is the vector $\begin{bmatrix} 25 \\ -139 \\ 63 \end{bmatrix}$ in its column space?

c) give the form of all 1x3 vectors in its row space.

d) Is the vector $[0 \quad 0 \quad 0]$ in its row space?

(9) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 9 \\ 13 \end{bmatrix}$, $C = \begin{bmatrix} -5 & -5 \\ -5 & -10 \\ -5 & -15 \end{bmatrix}$

a) Is $\mathcal{C}(A) \subset \mathcal{C}(B)$? Is $\mathcal{C}(AB) \subset \mathcal{C}(A)$?

If yes or no, explain why.

b) Is $\mathcal{R}(A) \subset \mathcal{R}(\mathcal{C})$? Is $\mathcal{R}(C) \subset \mathcal{R}(A)$?

(10) Let $A = \begin{bmatrix} 1 & 1 & -4 \\ 1 & 2 & -3 \\ 1 & 3 & -2 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{c}_{A1} & \underset{\sim}{c}_{A2} & \underset{\sim}{c}_{A3} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{r}_{A1}' \\ \underset{\sim}{r}_{A2}' \\ \underset{\sim}{r}_{A3}' \end{bmatrix}$

a) Give the rank(A).

b) Give a basis of the $\mathcal{C}(A)$

c) Give a basis for the $\mathcal{R}(A)$

d) Give x_1, x_2, x_3 s.t. $x_1 \underset{\sim}{r}_{a_1}' + x_2 \underset{\sim}{r}_{a_2}' + x_3 \underset{\sim}{r}_{a_3}' = \underset{\sim}{0}$ (Not $x_1 = x_2 = x_3 = 0$)

e) Give the ~~3rd row~~ row of A not used in your basis as a unique linear combination of the basis.

11) $\underset{m\times n}{A}, \underset{m\times p}{B}, \underset{q\times n}{C}$

a) Show that $\mathcal{C}(A) \subset \mathcal{C}(A, B)$ and $\mathcal{R}(A) \subset \mathcal{R}\begin{pmatrix} A \\ C \end{pmatrix}$

b) Show that if $\mathcal{C}(B) \subset \mathcal{C}(A)$ then $\mathcal{C}(A) = \mathcal{C}(A, B)$

12) Suppose $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

Give: $\text{rank}(A)$, $\text{rank}(B)$, $\text{rank}(C)$, $\text{rank}(A, B)$, $\text{rank}\begin{pmatrix} A \\ C \end{pmatrix}$

Show (for this case, though it holds in general):

$\text{rank}(A, B) \leq \text{rank}(A) + \text{rank}(B)$,

$\text{rank}\begin{pmatrix} A \\ C \end{pmatrix} \leq \text{rank}(A) + \text{rank}(C)$

13) Show that $tr(AB) = tr(BA)$ where $A \equiv m\times n$, $B \equiv n\times m$

14) Derive $tr(A'A)$ and show that $tr(A'A) = 0$ iff $A = 0$

15) Show for $\underset{m\times n}{A}, \underset{n\times p}{B}, \underset{n\times p}{C}$ that:

$AB = AC \Leftrightarrow A'AB = A'AC$

16) Consider vectors $\underset{\sim}{X} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\underset{\sim}{Y} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

Compute a) $\|\underset{\sim}{X}\|$, b) $\|\underset{\sim}{Y}\|$, c) $S(\underset{\sim}{X}, \underset{\sim}{Y})$, d) angle between $\underset{\sim}{X}, \underset{\sim}{Y}$

17 Consider matrices $A = \begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$

Compute: a) ~~Compute~~ $A \cdot B$ b) $\|A\|$ c) $\|B\|$, d) $\delta(A,B)$

e) $\cos\theta$ (Angle between A, B)

18 Give two matrices A, B s.t. $A \cdot B = 0$

19 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ obtain the QR decomposition of A

20 Let $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 9 & 0 \\ 3 & 10 & 3 \\ 1 & 6 & 3 \\ 0 & 4 & 0 \\ 0 & 3 & 1 \\ 7 & 34 & 8 \end{bmatrix}$ $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 7 & 3 \\ 21 & 12 \\ 32 & 16 \\ 22 & 10 \\ 8 & 4 \\ 9 & 4 \\ 99 & 49 \end{bmatrix}$

a) Show that $AX = B$

b) show that $\mathcal{C}(B) \subset \mathcal{C}(A)$ - Hint you have done this in a)

c) Show that $\underset{\sim}{k}'A = \underset{\sim}{0}'$ and $\underset{\sim}{k}'B = 0'$ for:

$\underset{\sim}{k}_1' = [1\ 1\ 1\ 1\ 1\ 1\ -1]$ $\underset{\sim}{k}_2' = [2.4\ 0.8\ -0.8\ 0\ -1\ 0\ 0]$

~~$\underset{\sim}{k}_2 = [2.4\ 0.8\ -0.8\ 0\ -1$~~

21 Let $A = \begin{bmatrix} 1 & 14 \\ 1 & 29 \\ 1 & 38 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} 79 \\ 57 \\ 28 \end{bmatrix}$

a) Show that there is not a solution to $A\underset{\sim}{X} = \underset{\sim}{b}$

b) Show that there is a solution to $A'A\underset{\sim}{X} = A'\underset{\sim}{b}$