Weighted Least Squares – Shotgun Spread – Summary of Results

Ordinary Least Squares

$$\mathbf{Y} = \mathbf{X\beta} + \boldsymbol{\varepsilon} \quad \boldsymbol{\varepsilon} \sim N\left(\mathbf{0}, \sigma^2 \mathbf{I}\right) \qquad \hat{\boldsymbol{\beta}} = \begin{bmatrix} -1.8518 \\ 0.6241 \end{bmatrix} \qquad s^2 = (3.228)^2 \qquad \hat{V}\left\{\hat{\boldsymbol{\beta}}\right\} = \begin{bmatrix} 1.9101 & -0.0521 \\ -0.0521 & 0.0017 \end{bmatrix}$$

```
> mod1.ols <- lm(Y1 ~ X1)
> summary(mod1.ols)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -1.85183    1.38207   -1.34    0.191
X1           0.62408    0.04167   14.98 6.79e-15 ***

Residual standard error: 3.228 on 28 degrees of freedom
Multiple R-squared:  0.889,     Adjusted R-squared:  0.8851
F-statistic: 224.3 on 1 and 28 DF,  p-value: 6.788e-15

> anova(mod1.ols)
Analysis of Variance Table
Response: Y1
          Df  Sum Sq Mean Sq F value    Pr(>F)
X1         1 2336.88 2336.88  224.29 6.788e-15 ***
Residuals 28  291.73   10.42

> vcov(mod1.ols)
            (Intercept)           X1
(Intercept)   1.9101208 -0.052094204
X1           -0.0520942  0.001736473
```

Estimated Weighted Least Squares with Weights = Reciprocal of Distance Sample Variance

$$\mathbf{Y} = \mathbf{X\beta} + \boldsymbol{\varepsilon} \qquad \boldsymbol{\varepsilon} \sim N\left(\mathbf{0}, \mathbf{V}\right) \qquad \mathbf{V} = \begin{bmatrix} \sigma_1^2\mathbf{I}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 \\ \mathbf{0}_6 & \sigma_2^2\mathbf{I}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 \\ \mathbf{0}_6 & \mathbf{0}_6 & \sigma_3^2\mathbf{I}_6 & \mathbf{0}_6 & \mathbf{0}_6 \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \sigma_4^2\mathbf{I}_6 & \mathbf{0}_6 \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \sigma_5^2\mathbf{I}_5 \end{bmatrix} \quad \hat{\mathbf{V}} = \begin{bmatrix} 0.210\mathbf{I}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 \\ \mathbf{0}_6 & 4.296\mathbf{I}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 \\ \mathbf{0}_6 & \mathbf{0}_6 & 10.136\mathbf{I}_6 & \mathbf{0}_6 & \mathbf{0}_6 \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & 9.133\mathbf{I}_6 & \mathbf{0}_6 \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & 28.886\mathbf{I}_5 \end{bmatrix}$$

$$\hat{\boldsymbol{\beta}}_W = \left(\mathbf{X}'\hat{\mathbf{V}}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\hat{\mathbf{V}}^{-1}\mathbf{Y} = \begin{bmatrix} -0.2190 \\ 0.5655 \end{bmatrix} \qquad s^2 = (1.006)^2 \qquad \hat{V}\left\{\hat{\boldsymbol{\beta}}_W\right\} = \begin{bmatrix} 0.1474 & -0.0098 \\ -0.0098 & 0.00084 \end{bmatrix}$$

```
> mod1.wls <- lm(Y1 ~ X1, weight=WT1)
> summary(mod1.wls)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.21903    0.38389  -0.571    0.573
X1           0.56553    0.02896  19.529   <2e-16 ***
```

```
Residual standard error: 1.006 on 28 degrees of freedom
Multiple R-squared:  0.9316,    Adjusted R-squared:  0.9292
F-statistic: 381.4 on 1 and 28 DF,  p-value: < 2.2e-16

> anova(mod1.wls)
Analysis of Variance Table
Response: Y1
          Df Sum Sq Mean Sq F value    Pr(>F)
X1         1 386.27  386.27  381.39 < 2.2e-16 ***
Residuals 28  28.36    1.01

> vcov(mod1.wls)
             (Intercept)            X1
(Intercept)  0.147371050 -0.0098243484
X1          -0.009824348  0.0008385934
```

Model With Variance Proportional to a Power of the Mean (mod1.gls)

$$E\{Y_i\} = \mu_i = \mathbf{x}_i'\boldsymbol{\beta} \quad \text{where } \mathbf{X} = \begin{bmatrix} \mathbf{x}_1' \\ \vdots \\ \mathbf{x}_n' \end{bmatrix} \qquad \sigma_i = \sigma\mu_i^{\delta} \quad V\{Y_i\} = \sigma^2\mu_i^{2\delta}$$

$$\hat{\boldsymbol{\beta}}_{\text{gls1}} = \begin{bmatrix} -0.0805 \\ 0.5483 \end{bmatrix} \qquad \hat{\delta} = 1.4571 \qquad \hat{\sigma} = 0.047045 \qquad \hat{V}\left\{\hat{\boldsymbol{\beta}}_{\text{gls1}}\right\} = \begin{bmatrix} 0.2096 & -0.01319 \\ -0.01319 & 0.00106 \end{bmatrix}$$

```
> mod1.gls <- gls(Y1 ~ X1, weights=varPower(form = ~fitted(.)), method="ML")
> summary(mod1.gls)
Generalized least squares fit by maximum likelihood
  Model: Y1 ~ X1
  Data: NULL
       AIC      BIC    logLik
  141.6413 147.2461 -66.82067

Variance function:
 Structure: Power of variance covariate
 Formula: ~fitted(.)
 Parameter estimates:
   power
1.457137

Coefficients:
                 Value Std.Error   t-value p-value
(Intercept) -0.0805454 0.4578215 -0.175932  0.8616
X1           0.5483329 0.0325164 16.863270  0.0000


Residual standard error: 0.04704754
Degrees of freedom: 30 total; 28 residual
> vcov(mod1.gls)
            (Intercept)            X1
(Intercept)  0.20960052 -0.013186985
X1          -0.01318699  0.001057316
```

Model With Separate Variance for Each Distance (mod3.gls) – Very similar to EWLS above

For Distance Group 1, $\sigma_i = \sigma$ For Distance Groups $j = 2,...,5$: $\sigma_i = \sigma m_j$

$$\hat{\boldsymbol{\beta}}_{\text{gls3}} = \begin{bmatrix} -0.2198 \\ 0.5663 \end{bmatrix} \quad \hat{\sigma} = 0.4202 \quad \hat{m}_2 = 4.92 \quad \hat{m}_3 = 7.66 \quad \hat{m}_4 = 6.63 \quad \hat{m}_5 = 12.91 \quad \hat{V}\left\{\hat{\boldsymbol{\beta}}_{\text{gls3}}\right\} = \begin{bmatrix} 0.1383 & -0.00947 \\ -0.00947 & 0.00082 \end{bmatrix}$$

```
> mod3.gls <- gls(Y1 ~ X1, weights=varIdent(form = ~ 1|rangeFireF), method="ML")
> summary(mod3.gls)
Generalized least squares fit by maximum likelihood
  Model: Y1 ~ X1
  Data: NULL
       AIC      BIC    logLik
  144.0713 153.8797 -65.03567

Variance function:
 Structure: Different standard deviations per stratum
 Formula: ~1 | rangeFireF
 Parameter estimates:
       10        20        30        40        50
 1.000000  4.924053  7.662055  6.628774 12.913290

Coefficients:
                 Value Std.Error   t-value p-value
(Intercept) -0.2197905 0.3718466 -0.591078  0.5592
X1           0.5662614 0.0286407 19.771220  0.0000

Residual standard error: 0.4201679
Degrees of freedom: 30 total; 28 residual

> vcov(mod3.gls)
             (Intercept)            X1
(Intercept)  0.138269886 -0.0094674149
X1          -0.009467415  0.0008202891
> anova(mod3.gls, mod1.gls)
         Model df      AIC      BIC    logLik   Test L.Ratio p-value
mod3.gls     1  7 144.0713 153.8797 -65.03567
mod1.gls     2  4 141.6413 147.2461 -66.82067 1 vs 2    3.57  0.3118
```

Comparison of Power Variance Model (mod1.gls) and Ordinary Least Squares (mod2.gls)

Variance as Power of Mean (Model 1): $V\{Y_i\} = \sigma^2 \mu_i^{2\delta}$ OLS (Model 2): $V\{Y_i\} = \sigma^2$

$AIC_1 = 141.64 \quad AIC_2 = 159.38 \quad BIC_1 = 147.25 \quad BIC_2 = 163.58$

$H_0: \delta = 0 \quad TS: X_{LR}^2 = -2(l_2 - l_1) = -2((-76.69) - (-66.82)) = 19.74 \quad P = P(\chi_{4-3}^2 \geq 19.74) < .0001$

```
> mod2.gls <- gls(Y1 ~ X1, method="ML")
> summary(mod2.gls)
Generalized least squares fit by maximum likelihood
  Model: Y1 ~ X1
  Data: NULL
      AIC      BIC   logLik
  159.375 163.5786 -76.6875
```

```
Coefficients:
                 Value Std.Error   t-value p-value
(Intercept) -1.8518333  1.382071 -1.339897  0.1911
X1           0.6240833  0.041671 14.976437  0.0000


Residual standard error: 3.118373
Degrees of freedom: 30 total; 28 residual

> anova(mod2.gls, mod1.gls)
         Model df      AIC      BIC    logLik   Test  L.Ratio p-value
mod2.gls     1  3 159.3750 163.5786 -76.68750
mod1.gls     2  4 141.6413 147.2461 -66.82067 1 vs 2 19.73367  <.0001
```

<u>Comparison of Power Variance Model (mod1.gls) and Heterogeneous by Distance (mod3.gls)</u>

$AIC_1 = 141.64 \quad AIC_3 = 144.07 \quad BIC_1 = 147.25 \quad BIC_3 = 153.88$

H_0 : Model 1 Provides Adequate Fit (Fewer Paramers by 3)

$TS: X_{LR}^2 = -2\left(l_1 - l_3\right) = -2\left(\left(-66.82\right) - \left(-65.04\right)\right) = 3.57 \quad P = P\left(\chi_{7-4}^2 \geq 3.57\right) = .3118$

```
> anova(mod3.gls, mod1.gls)
         Model df      AIC      BIC    logLik   Test L.Ratio p-value
mod3.gls     1  7 144.0713 153.8797 -65.03567
mod1.gls     2  4 141.6413 147.2461 -66.82067 1 vs 2    3.57  0.3118
```