Generalized Least Squares – Matrix Estimation – Yule-Walker Method

Model: $Y_t = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip} + \varepsilon_t \qquad \varepsilon_t + \rho_1\varepsilon_{t-1} + \rho_2\varepsilon_{t-2} + \cdots + \rho_q\varepsilon_{t-q} = \eta_t$

where $\{\varepsilon\} \perp \{\eta\}$. In matrix form:

$$\mathbf{Y} = \mathbf{X}\,\boldsymbol{\beta} + \boldsymbol{\varepsilon} \qquad V\{\boldsymbol{\varepsilon}\} = \boldsymbol{\Gamma}_n$$

1. Obtain Ordinary Least Squares Estimate of $\hat{\boldsymbol{\beta}}_{OLS}$ and Residual Vector (Labelled bols and e in program)

$$\hat{\boldsymbol{\beta}}_{0LS} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \qquad (p' \times 1) \qquad\qquad \mathbf{e} = \mathbf{Y} - \mathbf{X}\,\hat{\boldsymbol{\beta}}_{\mathbf{OLS}} \qquad (n \times 1)$$

2. Estimate the autocovariances up to lag q (Labelled gam_q in program)

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-h} e_t e_{t+h} \qquad h = 0, 1, 2, \ldots, q$$

3. Obtain the matrix of variances and autocovariances of the residuals to lag q, based on OLS residuals (Labelled gam2_q in program)

$$\hat{\boldsymbol{\Gamma}}_q = \begin{bmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) & \cdots & \hat{\gamma}(q-1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) & \cdots & \hat{\gamma}(q-2) \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\gamma}(q-1) & \hat{\gamma}(q-2) & \cdots & \hat{\gamma}(0) \end{bmatrix} \qquad (q \times q)$$

4. Obtain a column vector of autocovariances to lag q (Labelled gam1_q in program)

$$\hat{\gamma}_q = \begin{bmatrix} \hat{\gamma}(1) \\ \hat{\gamma}(2) \\ \vdots \\ \hat{\gamma}(q) \end{bmatrix} \qquad (q \times 1)$$

5. Obtain the vector of autoregressive parameter estimates based on the Yule-Walker equations (Labelled rho_q in the program)

$$\hat{\rho} = -\,\hat{\boldsymbol{\Gamma}}_q^{-1}\hat{\gamma}_q \qquad (q \times 1)$$

6. Obtain the estimate of σ^2, the variance of the independent random errors η_t (Labelled sigma2_q in the program)

$$\hat{\sigma}^2 = \hat{\gamma}(0) + \hat{\rho}'\hat{\gamma}_q$$

7. Obtain the upper triangular square root (Cholesky Decomposition, using the **root** function in SAS/IML) of $\hat{\boldsymbol{\Gamma}}_q^{-1}$ (Labelled p_q in program)

$$\hat{\boldsymbol{\Gamma}}_q^{-1} = \hat{\mathbf{P}}'_{\mathbf{q}}\hat{\mathbf{P}}_{\mathbf{q}} \qquad (q \times q)$$

8. Obtain the matrix $\hat{\mathbf{T}} = \hat{\mathbf{V}}^{-\mathbf{1/2}}$ (Labelled t_q) in program, it will be lower triangular)

$$\mathbf{T_{11}} = \sqrt{\hat{\sigma}^2}\hat{\mathbf{P}}_{\mathbf{q}} \qquad (q \times q)$$

$$\mathbf{T_{12}} = \mathbf{0} \qquad (q \times (n-q))$$

$$\mathbf{T_2} = \begin{bmatrix} \hat{\rho}_q & \hat{\rho}_{q-1} & \cdots & \hat{\rho}_1 & 1 & 0 & \cdots & 0 \\ 0 & \hat{\rho}_q & \cdots & \hat{\rho}_2 & \cdots \hat{\rho}_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \qquad ((n-q) \times n)$$

$$\hat{\mathbf{T}} = \begin{bmatrix} \mathbf{T_{11}} & & \mathbf{T_{12}} \\ & \mathbf{T_2} & \end{bmatrix} \qquad (n \times n)$$

The last array assumes that $n > 2(q+1)$ which it most certainly should be in practice.

9. Compute the estimates generalized least squares vector $\hat{\boldsymbol{\beta}}_{EGLS}$ (Labelled bgls_q in the program)

$$\hat{\boldsymbol{\beta}}_{EGLS} = (\mathbf{X}'\hat{\mathbf{T}}'\hat{\mathbf{T}}\mathbf{X})^{-\mathbf{1}}\mathbf{X}'\hat{\mathbf{T}}'\hat{\mathbf{T}}\mathbf{Y} \qquad (p' \times 1)$$

10. Estimate the error variance based on the transformed model (Labelled s2gls_q in the program)

$$\hat{\sigma}^2 = [(\mathbf{Y} - \mathbf{X}\ \hat{\boldsymbol{\beta}}_{EGLS})\hat{\mathbf{T}}'\hat{\mathbf{T}}(\mathbf{Y} - \mathbf{X}\ \hat{\boldsymbol{\beta}}_{EGLS})]/(n - p' - q)$$

Note: This denominator differs from the original authors' paper, they use $n - p'$

11. Obtain the (unscaled) variance-covariance matrix of $\hat{\boldsymbol{\beta}}_{EGLS}$ (Labelled c_q in the program)

$$(\mathbf{X}'\hat{\mathbf{T}}'\hat{\mathbf{T}}\mathbf{X})^{-\mathbf{1}} = (\mathbf{X}'\hat{\mathbf{V}}^{-\mathbf{1}}\mathbf{X})^{-\mathbf{1}} \qquad (p' \times p')$$

12. Obtain the estimated standard errors of $\hat{\boldsymbol{\beta}}_{EGLS}$ (Labelled sbgls_q in the program)

   - Multiply the the previous matrix by $\hat{\sigma}^2$
   - Obtain the square root of the diagonal elements

13. Obtain residual mean squares for estimates of autoregressive parameters ρ (Labelled s2rho_q in program)

$$s^2 = [\hat{\gamma}(0) + \hat{\rho}'\hat{\gamma}_q]/(n - p' - q)$$

Note: This denominator differs from the original authors' paper, they use $n - q$

14. Obtain the estimated standard error of ρ_i (Labelled serho_qi)

$$s\{\hat{\rho}_i\} = \sqrt{s^2\hat{\gamma}^{ii}} \qquad i = 1, \ldots, q$$

where $\hat{\gamma}^{ii}$ is the i^{th} diagonal element of $(\ \hat{\mathbf{\Gamma}})_q^{-1}$

15. Obtain the t-statistics for ρ_i $i = 1, \ldots, q$ (Labelled trho_qi and strung together into vectors trho_q)

$$t = \frac{\hat{\rho}_i}{s\{\hat{\rho}_i\}} \qquad i = 1, \ldots, q$$

16. These are compared with critical values of the t-distribution with $n - p' - q$

Source: Gallant, A.R. and J.J. Goebel,(1976),"Nonlinear Regression with Autocorrelated Errors",*JASA*,Vol. 71,Num.356,pp.961-967