(1)

$$Y_{ij} = \mu_i + \beta_j + \varepsilon_{ij} \qquad \underbrace{\beta_j \sim N(0,\sigma^2_\beta) \quad \varepsilon_{ij} \sim N(0,\sigma^2)}_{\text{Indep.}}$$

$$E\{Y_{ij}\} = E\{\mu_i + \beta_j + \varepsilon_{ij}\} = E\{\mu_i\} + E\{\beta_j\} + E\{\varepsilon_{ij}\} = \mu_i + 0 + 0 = \mu_i$$

$$V\{Y_{ij}\} = V\{\mu_i + \beta_j + \varepsilon_{ij}\} = \cancel{V\{\mu_i\}}^{0} + V\{\beta_j\} + V\{\varepsilon_{ij}\} + \cancel{2\,Cov\{\mu_i, \beta_j\}}^{0}$$
$$= \sigma^2_\beta + \sigma^2 \qquad + \cancel{2\,Cov\{\mu_i, \varepsilon_{ij}\}}^{0} + \cancel{2\,Cov\{\beta_j, \varepsilon_{ij}\}}^{0}$$

$$Y_{i\cdot} = \sum_{j=1}^{b} Y_{ij} = Y_{i1} + \cdots + Y_{ib} \qquad Cov\{Y_{ij}, Y_{ij'}\} = \begin{cases} \sigma^2_\beta + \sigma^2 & j = j' \\ 0 & j \neq j' \end{cases}$$

$$Cov\{Y_{i1}, Y_{i2}\} = Cov\{\mu_i + \beta_1 + \varepsilon_{i1},\ \mu_i + \beta_2 + \varepsilon_{i2}\} \qquad Cov\{Y_{ij}, Y_{i'j}\} = \sigma^2_\beta \ \ i \neq i'$$

$$V\{Y_{i\cdot}\} = V\{Y_{i1} + \cdots + Y_{ib}\} = \sum_j V\{Y_{ij}\} + 2\sum_{j<j'} Cov\{Y_{ij}, Y_{ij'}\}$$
$$V\{\bar{Y}_{i\cdot}\} = V\{\tfrac{1}{b} Y_{i\cdot}\} = \left(\tfrac{1}{b}\right)^2 b(\sigma^2_\beta + \sigma^2) = \tfrac{1}{b}(\sigma^2_\beta + \sigma^2) = V\{\bar{Y}_{i\cdot}\}$$

(2)

$$Cov\{Y_{i\cdot}, Y_{i'\cdot}\} = Cov\{Y_{i1} + \cdots + Y_{ib},\ Y_{i'1} + \cdots + Y_{i'b}\}$$
$$= Cov\{Y_{i1}, Y_{i'1}\}^{\sigma^2_\beta} + Cov\{Y_{i1}, Y_{i'2}\}^{0} + \cdots + Cov\{Y_{i1}, Y_{i'b}\}^{0} +$$
$$\vdots$$
$$Cov\{Y_{ib}, Y_{i'1}\}^{0} + Cov\{Y_{ib}, Y_{i'2}\}^{0} + \cdots + Cov\{Y_{ib}, Y_{i'b}\}^{\sigma^2_\beta}$$
$$= b\sigma^2_\beta$$

$$Cov\{\bar{Y}_{i\cdot}, \bar{Y}_{i'\cdot}\} = Cov\{\tfrac{1}{b} Y_{i\cdot}, \tfrac{1}{b} Y_{i'\cdot}\}$$
$$= \left(\tfrac{1}{b}\right)^2 Cov\{Y_{i\cdot}, Y_{i'\cdot}\} = \left(\tfrac{1}{b}\right)^2 b\,\sigma^2_\beta = \frac{\sigma^2_\beta}{b}$$

$$V\{\bar{Y}_{i\cdot} - \bar{Y}_{i'\cdot}\} = (1)^2 V\{\bar{Y}_{i\cdot}\} + (-1)^2 V\{\bar{Y}_{i'\cdot}\} + 2(1)(-1)\,Cov\{\bar{Y}_{i\cdot}, \bar{Y}_{i'\cdot}\}$$
$$= 2\,\frac{\sigma^2_\beta + \sigma^2}{b} - 2\,\frac{\sigma^2_\beta}{b} = 2\,\frac{\sigma^2}{b} \qquad \hat{SE}\{\bar{Y}_{i\cdot} - \bar{Y}_{i'\cdot}\} = \sqrt{\frac{2MSE}{b}}$$