Chapter 9 - Factor Analysis

Goal: Discover m unobservable, underlying Factors based on Covariances / Correlations among a set of p observed responses.

9.2 - orthogonal Factor Model

$$\underset{\sim}{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_p \end{bmatrix} \quad w/ \quad E\{\underset{\sim}{X}\} = \underset{\sim}{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_p \end{bmatrix} \quad V\{X\} = \Sigma$$

$$X_1 - \mu_1 = l_{11} F_1 + \cdots + l_{1m} F_m + \varepsilon_1$$

$$\vdots$$

$$X_p - \mu_p = l_{p1} F_1 + \cdots + l_{pm} F_m + \varepsilon_p$$

Matrix form: $\underbrace{\underset{\sim}{X} - \underset{\sim}{\mu}}_{p\times 1} = \underset{p\times m}{L} \; \underset{m\times 1}{F} + \underset{p\times 1}{\varepsilon}$

$F_1, \ldots, F_m \equiv$ common factors

$l_{ij} \equiv$ Loading of i^{th} variable on the j^{th} factor

$\varepsilon_i \equiv$ errors, specific factors

$L \equiv$ matrix of factor loadings

<u>Assumptions regarding random $\underset{\sim}{F}, \underset{\sim}{\varepsilon}$</u>

$$E\{\underset{\sim}{F}\} = \underset{m\times 1}{\underset{\sim}{0}} \qquad Cov\{F\} = E\{FF'\} = \underset{m\times m}{I}$$

$$E\{\underset{\sim}{\varepsilon}\} = \underset{p\times 1}{\underset{\sim}{0}} \qquad Cov\{\underset{\sim}{\varepsilon}\} = E\{\underset{\sim}{\varepsilon}\underset{\sim}{\varepsilon}'\} = \Psi = \begin{bmatrix} \psi_1 & & 0 \\ & \psi_2 & \\ 0 & & \ddots \; \psi_p \end{bmatrix}$$

$$Cov\{\varepsilon, F\} = E\{\underset{\sim}{\varepsilon} F'\} = \underset{p\times m}{\underset{\sim}{0}}$$

$$(\underset{\sim}{X}-\underset{\sim}{\mu})(\underset{\sim}{X}-\underset{\sim}{\mu})' = (LF+\varepsilon)(LF+\varepsilon)' = (LF+\varepsilon)((LF)'+\varepsilon')$$

$$= LFF'L' + LF\varepsilon' + \varepsilon F'L' + \varepsilon\varepsilon'$$

$$\Sigma = E\{(X-\mu)(X-\mu)'\} = E\{LFF'L'\} + E\{L\underset{\sim}{F}\underset{\sim}{\varepsilon}'\} + E\{\varepsilon F'L'\} + E\{\varepsilon\varepsilon'\}$$

$$= LL' + \Psi$$

$$(X-\mu)F' = (LF+\varepsilon)F' = LFF' + \varepsilon F'$$

$$\Rightarrow Cov\{X, F\} = E\{(X-\mu)F'\} = L\,E\{FF'\} + E\{\varepsilon F'\} = L$$

$$\underset{p\times m}{L} = \begin{bmatrix} l_{11} & \cdots & l_{1m} \\ \vdots & & \\ l_{p1} & \cdots & l_{pm} \end{bmatrix} \Rightarrow \underset{p\times p}{LL'} = \begin{bmatrix} \sum_{j=1}^{m} l_{1j}^2 & \cdots & \sum_{j=1}^{m} l_{1j}l_{pj} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{m} l_{pj}l_{1j} & \cdots & \sum_{j=1}^{m} l_{pj}^2 \end{bmatrix}$$

$$\Rightarrow V\{X_i\} = \sigma_{ii} = \sum_{j=1}^{m} l_{ij}^2 + \Psi_i$$

$$i \neq k: \; Cov\{X_i, X_k\} = \sum_{j=1}^{m} l_{ij}l_{kj}$$

$$Cov\{X_i, F_j\} = l_{ij}$$

$$\underset{V\{X_i\}}{\sigma_{ii}} = \underbrace{l_{i1}^2 + \cdots + l_{im}^2}_{\text{Communality}} + \underbrace{\varepsilon_i^2}_{\substack{\text{specific variance} \\ \text{aka uniqueness}}}$$

EXAMPLE

$$\Sigma = \begin{bmatrix} 38 & 23 & -9 & 11 \\ 23 & 37 & 9 & 1 \\ -9 & 9 & 17 & -7 \\ 11 & 1 & -7 & 7 \end{bmatrix}$$

$$L = \begin{bmatrix} 6 & 1 \\ 3 & 5 \\ -2 & 3 \\ 2 & -1 \end{bmatrix} \qquad LL' = \begin{bmatrix} 37 & 23 & -9 & 11 \\ 23 & 34 & 9 & 1 \\ -9 & 9 & 13 & -7 \\ 11 & 1 & -7 & 5 \end{bmatrix} \quad \Psi = \mathrm{diag}\begin{Bmatrix} 1 \\ 3 \\ 4 \\ 2 \end{Bmatrix}$$

Commonality of X_1 : $h_1^2 = l_{11}^2 + l_{12}^2 = 6^2 + 1^2 = 37$

$$\sigma_{11} = h_1^2 + \psi_1 = l_{11}^2 + l_{12}^2 + \psi_1 = 37 + 1 = 38$$

$$X \equiv p + \frac{p(p-1)}{2} = \frac{2p + p^2 - p}{2} = \frac{p(p+1)}{2}$$ variances and covariances

$\Rightarrow$ pm Factor loadings $\{l_{ij}\}$ + p specific variances $\{\psi_i\}$

$p = m \Rightarrow \Sigma = LL'$, $\Psi = 0$ matrix

Most useful when m is small relative to p.

Most Σ cannot be factored as $LL' + \Psi$ for m much less than p.

Example - $p=3, \quad m=1 \quad \Sigma = \begin{bmatrix} 1 & -.5 & .8 \\ -.5 & 1 & .2 \\ .8 & .2 & 1 \end{bmatrix}$

$$F = F_1 \qquad L = \begin{bmatrix} l_{11} \\ l_{21} \\ l_{31} \end{bmatrix} \qquad LL' = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 & l_{21}l_{31} \\ l_{31}l_{11} & l_{31}l_{21} & l_{31}^2 \end{bmatrix} \quad \Psi = \begin{bmatrix} \psi_1 & 0 & 0 \\ 0 & \psi_2 & 0 \\ 0 & 0 & \psi_3 \end{bmatrix}$$

$$\Rightarrow \quad 1 = l_{11}^2 + \psi_1 \qquad -.5 = l_{11}l_{21} \qquad .8 = l_{11}l_{31}$$

$$1 = l_{21}^2 + \psi_2 \qquad .2 = l_{21}l_{31}$$

$$1 = l_{31}^2 + \psi_3$$

$$.8 = l_{11}l_{31}$$
$$.2 = l_{21}l_{31}$$

$$\Rightarrow 4 = \frac{l_{11}}{l_{21}} \Rightarrow l_{11} = 4 l_{21} \qquad -.5 = l_{11}l_{21} = 4l_{21}l_{21} = 4l_{21}^2$$

$\Rightarrow l_{21}^2 = -\frac{1}{8}$ Not possible for real l_{21}

$m > 1 \Rightarrow$ ambiguity in Factor Model

$T =$ $m \times m$ orthogonal matrix $\Rightarrow TT' = T'T = I$

$$X - \mu = LF + \varepsilon = LTT'F + \varepsilon = L^*F^* + \varepsilon \qquad \begin{array}{l} L^* = LT \\ F^* = T'F \end{array}$$

$$E\{F^*\} = T'E\{F\} = 0 \qquad V\{F^*\} = T'IT = T'T = I$$

$\Rightarrow L, L^*$ undistinguishable

$$\Sigma = LL' + \Psi = L^*L^{*\prime} + \Psi$$

(Rationale for Factor Rotations)

9.3. Estimation Methods

Data $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n$ w/ $\underset{\sim}{x}_j = \begin{bmatrix} x_{j1} \\ \vdots \\ x_{jp} \end{bmatrix} \to S = \frac{1}{n-1} \sum_{j=1}^{n} (x_j - \bar{x})(x_j - \bar{x})'$ and R

If off-diagonal elements of S and R are virtually 0, FA will not be useful.

Principal Component (Principal Factor) Method

Spectral Decomposition: $\Sigma = \lambda_1 \underset{\sim}{e}_1 \underset{\sim}{e}_1' + \ldots + \lambda_p \underset{\sim}{e}_p \underset{\sim}{e}_p'$ λ_i = eigvals, $\underset{\sim}{e}_i$ = eigvecs

$$= \left[\sqrt{\lambda_1}\, \underset{\sim}{e}_1 : \cdots : \sqrt{\lambda_p}\, \underset{\sim}{e}_p\right] \begin{bmatrix} \sqrt{\lambda_1}\, \underset{\sim}{e}_1' \\ \vdots \\ \sqrt{\lambda_p}\, \underset{\sim}{e}_p' \end{bmatrix} \qquad (m = p) \quad \underset{\sim}{\varepsilon} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \underset{p \times p}{\Sigma} = \underset{p \times p}{L} \underset{p \times p}{L'} + \underset{p \times p}{0} = LL'$$

$\Rightarrow$ Factor loading on Factor j are $\sqrt{\lambda_j}\, \underset{\sim}{e}_j$

Not useful since as many factors as variables.

Last $p-m$ eigenvalues are small $\Rightarrow$

$$\Sigma \doteq \left[\sqrt{\lambda_1}\, \underset{\sim}{e}_1 \ \ldots \ \sqrt{\lambda_m}\, \underset{\sim}{e}_m\right] \begin{bmatrix} \sqrt{\lambda_1}\, \underset{\sim}{e}_1' \\ \vdots \\ \sqrt{\lambda_m}\, \underset{\sim}{e}_m' \end{bmatrix} = \underset{p \times m}{L} \underset{m \times p}{L'}$$

Variances of specific factors can be obtained as diagonal elements of $\Sigma - LL'$

$\Sigma \doteq LL' + \Psi \qquad \psi_i = \sigma_{ii} - \sum_{j=1}^{m} l_{ij}^2 \quad i=1,\dots,p$

Use centred data: $\underset{\sim}{x}_j - \bar{\underset{\sim}{x}} \quad j=1,\dots,n$ (same covariance matrix S)

Standardized variables: when units are very different

$$\underset{\sim}{z}_j = \begin{bmatrix} \frac{x_{j1}-\bar{x}_1}{\sqrt{s_{11}}} \\ \vdots \\ \frac{x_{jp}-\bar{x}_p}{\sqrt{s_{pp}}} \end{bmatrix} \quad j=1,\dots,n$$

(Covariance matrix of $\underset{\sim}{z}_j$ is R).

Principal Component Solution

$$S = \hat{\lambda}_1 \hat{\underset{\sim}{e}}_1 \hat{\underset{\sim}{e}}_1' + \dots + \hat{\lambda}_p \hat{\underset{\sim}{e}}_p \hat{\underset{\sim}{e}}_p'$$

~~Let~~ $m < p \equiv$ # of common factors

$$\tilde{L} = \left[\sqrt{\hat{\lambda}_1}\, \hat{\underset{\sim}{e}}_1 \ \cdots \ \sqrt{\hat{\lambda}_m}\, \hat{\underset{\sim}{e}}_m\right]$$

$$\tilde{\Psi} = \begin{bmatrix} \tilde{\psi}_1 & & 0 \\ & \tilde{\psi}_2 & \\ 0 & & \ddots \ \tilde{\psi}_p \end{bmatrix} \quad \text{w/} \quad \tilde{\psi}_i = s_{ii} - \sum_{j=1}^{m} \tilde{l}_{ij}^2$$

w/ communalities $\tilde{h}_i^2 = \tilde{l}_{i1}^2 + \dots + \tilde{l}_{im}^2$

$l_{ij} = \sqrt{\hat{\lambda}_j}\, e_{ij}$

Problem: Choosing m.

$S - (\tilde{L}\tilde{L}' + \tilde{\Psi})$ has 0's on diagonal and "residuals" in off diagonal cells.

Sum of squared elements of $\left(S - (\tilde{L}\tilde{L}' + \tilde{\Psi})\right) \leq \hat{\lambda}_{m+1}^2 + \cdots + \hat{\lambda}_p^2$

$\Rightarrow$ Small value for the sum of squares of removed eigenvalues $\Rightarrow$ small value for sum of squared errors of approximation ("residuals").

Proportion of Total Sample variance due to j^{th} Factor $= \begin{cases} \dfrac{\hat{\lambda}_j}{s_{11} + \cdots + s_{pp}} & \text{for FA from } S \\ \dfrac{\hat{\lambda}_j}{p} & \text{for FA from } R \end{cases}$

Maximum Likelihood Estimation

F_j, $\varepsilon_j \sim$ Normal $\qquad (\underset{\sim}{X}_j - \underset{\sim}{\mu}) = L F_j + \varepsilon_j \sim$ Normal

$$L(\mu, \Sigma) = (2\pi)^{-\frac{np}{2}} |\Sigma|^{-n/2} \times \exp\left\{-\frac{1}{2} tr\left(\Sigma^{-1}\left[\sum_j (\underset{\sim}{x}_j - \underset{\sim}{\bar{x}})(\underset{\sim}{x}_j - \underset{\sim}{\bar{x}})' + n(\underset{\sim}{\bar{x}} - \underset{\sim}{\mu})(\bar{x} - \mu)'\right]\right)\right\}$$

$$= (2\pi)^{-\frac{(n-1)p}{2}} |\Sigma|^{-\frac{(n-1)}{2}} \exp\left\{-\frac{1}{2} tr\left(\Sigma^{-1}\left(\sum_j (x_j - \bar{x})(x_j - \bar{x})'\right)\right)\right\} \times$$
$$(2\pi)^{-p/2} |\Sigma|^{-1/2} \exp\left\{-\frac{n}{2} (\bar{x} - \mu)' \Sigma^{-1} (\bar{x} - \mu)\right\}$$

where $\Sigma = LL' + \Psi$ $\qquad$ U~~niqueness Cond~~

Uniqueness Condition: $L' \Psi^{-1} L \equiv \Delta$ (Diagonal matrix).

Result 9.1 $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n \equiv$ random sample $\sim N_p(\underset{\sim}{\mu}, \Sigma)$

$\Sigma = LL' + \Psi \equiv$ Covariance matrix for m common factor model.

Maximum Likelihood Estimators $\hat{L}, \hat{\Psi}, \hat{\underset{\sim}{\mu}} = \bar{\underset{\sim}{x}}$ maximize normal joint density s.t. $\hat{L}'\hat{\Psi}^{-1}\hat{L} \equiv$ Diagonal.

MLE's for communalities: $\hat{h}_i^2 = \hat{l}_{i1}^2 + \hat{l}_{i2}^2 + \cdots + \hat{l}_{im}^2 \quad i=1,\ldots,p$

$\Rightarrow$ Proportion of total sample variance due to j^{th} factor:

$$\frac{\hat{l}_{1j}^2 + \cdots + \hat{l}_{pj}^2}{s_{11} + \cdots + s_{pp}}$$ (Invariance property of MLE's)

Correlation Matrix Form

$$\rho = V^{-1/2}\Sigma V^{-1/2} = (V^{-1/2}L)(V^{-1/2}L)' + V^{-1/2}\Psi V^{-1/2}$$

MLE: $$\hat{\rho} = (\hat{V}^{-1/2}\hat{L})(\hat{V}^{-1/2}\hat{L})' + \hat{V}^{-1/2}\hat{\Psi}\hat{V}^{-1/2} = \hat{L}_z\hat{L}_z' + \hat{\Psi}_z$$

MLE of communalities: $\hat{h}_i^2 = \hat{l}_{i1}^2 + \cdots + \hat{l}_{im}^2 \quad i=1,\ldots,p$

Proportion of total (standardized) sample variance due to j^{th} factor:

$$\frac{\hat{l}_{1j}^2 + \cdots + \hat{l}_{pj}^2}{p}$$ $\hat{l}_{ij} \equiv (i,j)^{th}$ element of $\hat{L}_z$

Equivalence of Factoring S and R

$\hat{l}_{ij} = \hat{l}_{z,ij}\sqrt{\hat{\sigma}_{ii}} \qquad \hat{\Psi}_i = \hat{\Psi}_{z,i}\,\hat{\sigma}_{ii}$ ($\hat{\sigma}_{ii}$ based on n as divisor)

Large-Sample Test for # of Common Factors (m)

$$H_0: \underset{p\times p}{\Sigma} = \underset{p\times m}{L}\underset{m\times p}{L'} + \underset{p\times p}{\Psi} \qquad H_A: \Sigma \equiv \text{any other pos. def. matrix}$$

$$\hat{\Sigma} = \hat{L}\hat{L}' + \hat{\Psi} \qquad S_n = \frac{1}{n}\sum_{j=1}^{n}(x_j - \bar{x})(x_j - \bar{x})'$$

Bartlett's Corrected Test Statistic: $\left(n-1-\left(\frac{2p+4m+5}{6}\right)\right)\ln\frac{|\hat{\Sigma}|}{|S_n|}$

Reject H_0 if $TS: > \chi^2_{[(p-m)^2-p-m]/2}(\alpha)$

Note: $\dfrac{|\hat{\Sigma}|}{|S_n|} = \dfrac{|\hat{L}_z\hat{L}_z' + \hat{\Psi}_z|}{|R|}$

9.4 Factor Rotation

$\hat{L} \equiv p\times m$ matrix of estimated factor loadings

Rotated Loadings: $\hat{L}^* = \hat{L}T$ w/ $TT' = T'T = I$

$\Rightarrow$ Estimated variance/correlation matrix is unchanged.

Goal: Each variable loads high on one factor w/ small/moderate loadings on other factors.

In the case for 2 factors (or 2-at-a-time).

Clockwise $\Rightarrow T = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$ Counter-Clockwise $\Rightarrow T = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

Varimax Rotation

$$\tilde{\ell}^*_{ij} = \frac{\hat{\ell}^*_{ij}}{\hat{h}_i} = \text{rotated coefficients scaled by } \sqrt{\ } \text{ of communalities}$$

Varimax procedure selects T to maximize:

$$V = \frac{1}{p} \sum_{j=1}^{m} \left[\sum_{i=1}^{p} \tilde{\ell}^{*4}_{ij} - \left(\sum_{i=1}^{p} \tilde{\ell}^{*2}_{ij} \right)^2 \Big/ p \right]$$

$$V \propto \sum_{j=1}^{m} (\text{variance of squared scaled loadings for factor } j)$$

Procedure tries to spread out squares of loadings of each factor as much as possible. (Large vs small).

Factor Scores

Estimates of values for the unobserved random factors F_j
$j = 1, \ldots, n$

$\hat{f}_j$ estimates f_j attained by F_j (j^{th} case)

2 approaches: Weighted Least Squares, Regression.

Both approaches:

1) Treat $\hat{\ell}_{ij}$ and $\tilde{\psi}_i$ as "true" values ℓ_{ij}, ψ_i
2) Are linear transformations of original data and can be made to rotated (usually) and unrotated loadings.

Weighted Least Squares

weights ≡ reciprocal of $V\{\varepsilon_i\} = \psi_i$

$$\sum_{i=1}^{p} \frac{\varepsilon_i^2}{\psi_i} = \underset{\sim}{\varepsilon}'\Psi^{-1}\underset{\sim}{\varepsilon} = (X-\mu-Lf)'\Psi^{-1}(X-\mu-Lf)$$

$$\Rightarrow \hat{f} = (L'\Psi^{-1}L)^{-1}L'\Psi^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu})$$

Replacing unknowns w/ estimates, factor scores for j^{th} case:

$$\hat{f}_j = (\hat{L}'\hat{\Psi}^{-1}\hat{L})^{-1}\hat{L}'\hat{\Psi}^{-1}(\underset{\sim}{x}_j-\underset{\sim}{\bar{x}})$$

For ML estimates, uniqueness condition $\Rightarrow \hat{L}'\hat{\Psi}^{-1}\hat{L} = \hat{\Delta}$ (diagonal)

S factored: $\hat{f}_j = (\hat{L}'\hat{\Psi}^{-1}\hat{L})^{-1}\hat{L}'\hat{\Psi}^{-1}(\underset{\sim}{x}_j-\underset{\sim}{\bar{x}}) = \hat{\Delta}^{-1}\hat{L}'\hat{\Psi}^{-1}(x_j-\bar{x}) \quad j=1,\dots,n$

R factored: $\hat{f}_j = (\hat{L}_z'\hat{\Psi}_z^{-1}\hat{L}_z)^{-1}\hat{L}_z'\hat{\Psi}_z^{-1}\underset{\sim}{z}_j = \hat{\Delta}_z^{-1}\hat{L}_z'\hat{\Psi}_z^{-1}\underset{\sim}{z}_j \quad j=1,\dots,n$

$$z_j = D^{-1/2}(\underset{\sim}{x}_j-\underset{\sim}{\bar{x}}) \qquad D=\begin{bmatrix} s_{11} & & 0 \\ & s_{22} & \\ 0 & & \ddots\ s_{pp}\end{bmatrix} \qquad \hat{\rho} = \hat{L}_z\hat{L}_z' + \hat{\Psi}_z$$

When rotated loadings are used, $\hat{L}^* = \hat{L}T \Rightarrow \hat{f}_j^* = T'\hat{f}_j$

(*) Estimates Based on Principal components, typically:

S: $\hat{f}_j = (\tilde{L}'\tilde{L})^{-1}\tilde{L}'(\underset{\sim}{x}_j-\bar{x})$ R: $\hat{f}_j = (\tilde{L}_z'\tilde{L}_z)^{-1}\tilde{L}_z'\underset{\sim}{z}_j$

where $\tilde{L} = [\sqrt{\hat{\lambda}_1}\,\hat{\underset{\sim}{e}}_1 \cdots \sqrt{\hat{\lambda}_m}\,\underset{\sim}{e}_m] \Rightarrow \hat{f}_j = \begin{bmatrix} \frac{1}{\sqrt{\hat{\lambda}_1}}\underset{\sim}{e}_1'(x_j-\bar{x}) \\ \vdots \\ \frac{1}{\sqrt{\hat{\lambda}_m}}\underset{\sim}{e}_m'(\underset{\sim}{x}_j-\bar{x}) \end{bmatrix}$

$\Rightarrow \frac{1}{n}\sum_{j=1}^{n}\hat{f}_j = 0 \qquad \frac{1}{n-1}\sum_{j=1}^{n}\hat{f}_j\hat{f}_j' = I$

Regression Method

$X - \mu = LF + \varepsilon$ Treating L, Ψ as known

$F, \varepsilon \sim$ Normal w/ $E\{F\}=0 \quad V\{F\}=I \quad E\{\varepsilon\}=0 \quad V\{\varepsilon\}=\Psi$

$\Rightarrow X - \mu = LF + \varepsilon \sim N_p(0, LL' + \Psi)$

Joint distribution of $(X-\mu), F \sim N_{m+p}(0, \Sigma^*)$

where: $\underset{(m+p)\times(m+p)}{\Sigma^*} = \left[\begin{array}{c|c} \Sigma = LL' + \Psi & L \\ \hline L' & I \end{array}\right]$ (From model $Cov\{X, F\} = L$)

$E\{\underline{F}|\underline{x}\} = L'\Sigma^{-1}(\underline{x} - \underline{\mu}) = L'(LL'+\Psi)^{-1}(\underline{x} - \underline{\mu})$

$V\{\underline{F}|\underline{x}\} = I - L'\Sigma^{-1}L = I - L'(LL'+\Psi)^{-1}L$

$\Rightarrow \hat{f}_j = \hat{L}'\hat{\Sigma}^{-1}(\underline{x}_j - \underline{\bar{x}}) = \hat{L}'(\hat{L}\hat{L}' + \hat{\Psi})^{-1}(\underline{x}_j - \underline{\bar{x}}) = \hat{f}_j^R$

Note: $\hat{L}'(\hat{L}\hat{L}' + \hat{\Psi})^{-1} = (I + \hat{L}'\hat{\Psi}^{-1}\hat{L})^{-1}\hat{L}'\hat{\Psi}^{-1}$

WLS Estimate $\hat{f}_j^{LS} = (\hat{L}'\hat{\Psi}^{-1}\hat{L})^{-1}(I + \hat{L}\hat{\Psi}^{-1}\hat{L})\hat{f}_j^R = (I + (\hat{L}\hat{\Psi}^{-1}\hat{L})^{-1})\hat{f}_j^R$

ML Estimates $\Rightarrow (\hat{L}'\hat{\Psi}^{-1}\hat{\Psi})^{-1} = \hat{\Delta}^{-1}$ (diagonal)

If elements of $\hat{\Delta}$ small, WLS and Regression give similar factor scores.

Typically practitioners replace $\hat{\Psi}$ with S.

$S \Rightarrow \hat{f}_j = \hat{L}' S^{-1} (\underset{\sim}{x}_j - \bar{\underset{\sim}{x}}) \quad j = 1, \ldots, n$

$R \Rightarrow \hat{f}_j = \hat{L}_z' R^{-1} \underset{\sim}{z}_j \quad z_j = D^{-1/2} x_j - \bar{x} \quad \hat{\rho} = \hat{L}_z \hat{L}_z' + \hat{\Psi}.$

Rotated Loadings $\Rightarrow \hat{f}_j^{*} = T' \hat{f}_j$

Number of factors to be extracted: subject to:

$(p-m)^2 \geq p+m \quad \left(m < \frac{1}{2}\left(2p+1-\sqrt{8p+1}\right) \right)$

Problems: 1, 2, 3, 6, 8, 10, 12, 13, 19a-d, 24