Chapter 3 Sample Geometry and Random Sampling

$$\underset{n\times p}{X} = \begin{bmatrix} x_{11} & \cdots & x_{1p} \\ x_{21} & \cdots & x_{2p} \\ \vdots & & \\ x_{n1} & \cdots & x_{np} \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$$

n vectors – p-dim space

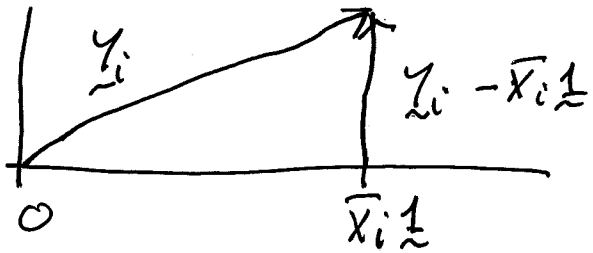
$x_i' = [x_{i1} \cdots x_{ip}]$

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_p \end{bmatrix}$$

Computing the mean vector: $\bar{x} = \frac{1}{n} X' \underset{\sim}{1}_n$

$X = [\underset{\sim}{y}_1 ; \underset{\sim}{y}_2 ; \cdots ; \underset{\sim}{y}_p]$ p vectors in n-dim space.

$$\underset{\sim}{y}_i' \left(\frac{1}{\sqrt{n}}\underset{\sim}{1}\right)\left(\frac{1}{\sqrt{n}}\underset{\sim}{1}\right) = \frac{\sum_{j=1}^{n} x_{ji}}{n}\underset{\sim}{1} = \bar{x}_i \underset{\sim}{1}_n$$



$$\underset{\sim}{d}_i = \underset{\sim}{y}_i - \bar{x}_i\underset{\sim}{1} = \begin{bmatrix} x_{1i} - \bar{x}_i \\ x_{2i} - \bar{x}_i \\ \vdots \\ x_{ni} - \bar{x}_i \end{bmatrix} \qquad i = 1, \ldots, p$$

Note: Book uses distance D

Let $D = [d_1 ; d_2 ; \cdots ; d_p]$

$$D'D = \begin{bmatrix} \sum_j (x_{j1} - \bar{x}_1)^2 & \sum_j (x_{j1} - \bar{x}_1)(x_{j2} - \bar{x}_2) & \cdots & \sum_j (x_{j1} - \bar{x}_1)(x_{jp} - \bar{x}_p) \\ \sum_j (x_{j1} - \bar{x}_1)(x_{jp} - \bar{x}_p) & \sum_j (x_{j2} - \bar{x}_2)(x_{jp} - \bar{x}_p) & \cdots & \sum_j (x_{jp} - \bar{x}_p)^2 \end{bmatrix}$$

$$S_n = \frac{1}{n} D'D \qquad V^{1/2} = \text{sqrt}(\text{diag}(S_n))$$

$$R = V^{-1/2} S_n V^{-1/2}$$

(Equation 2-6)

$$d_i' d_k = \sum_j (x_{ji} - \bar{x}_i)(x_{jk} - \bar{x}_k) = L_{d_i} L_{d_k} \cos\theta_{ik}$$

$$\Rightarrow \sum_j (x_{ji} - \bar{x}_i)(x_{jk} - \bar{x}_k) = \sqrt{\sum_j (x_{ji} - \bar{x}_i)^2} \sqrt{\sum_j (x_{ki} - \bar{x}_k)^2} \cos(\theta_{ik})$$

$$\Rightarrow r_{ik} = \frac{s_{ik}}{\sqrt{s_{ii}}\sqrt{s_{kk}}} = \cos\theta_{ik}$$

θ_{ik} is angle formed by vectors $\underset{\sim}{d_i}$, $\underset{\sim}{d_k}$

Geometrical Interpretation of the Sample

(Column of k)

1) Projection of y_i onto vector $\underset{\sim}{1}$ is $\bar{x}_i \underset{\sim}{1}$

Length of $\bar{x}_i 1 = \sqrt{\bar{x}_i 1' 1 \bar{x}_i} = |\bar{x}_i|\sqrt{n}$

2) $S_n = \frac{1}{n} D'D$ $\qquad D = [d_1 ; d_2 \cdots ; d_p]$ (Deviations)

$d_i = \underset{\sim}{y_i} - \bar{x}_i \underset{\sim}{1}_n$ $\qquad (L_{d_i})^2 = n s_{ii}$ $\qquad L_{d_i} L_{d_k} = n s_{ik} = d_i' d_k$

3) $r_{ik} =$ Cos of angle between $\underset{\sim}{d_i}$, $\underset{\sim}{d_k}$.

3.3 Random Samples and $E\{\bar{X}\}$ and $E\{S_n\}$

$$X = \begin{bmatrix} X_{11} & \cdots & X_{1p} \\ \vdots & \ddots & \vdots \\ X_{n1} & \cdots & X_{np} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{X}_1' \\ \vdots \\ \underset{\sim}{X}_n' \end{bmatrix} \qquad \underset{\sim}{X}_i' = [X_{i1} \cdots X_{ip}]$$

Row vectors independent observations from common

$f(\underset{\sim}{x}) = f(x_1, \ldots, x_p)$ $\Rightarrow$ $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ are <u>random sample</u> from $f(\underset{\sim}{x})$

$\Rightarrow$ joint density is $f(\underset{\sim}{x}_1) f(\underset{\sim}{x}_2) \cdots f(\underset{\sim}{x}_n)$ $f(\underset{\sim}{x}_j) = f(x_{j1}, \ldots, x_{jp})$

1) measurements of observation in a single trial

$\underset{\sim}{X}_j' = [X_{j1}, \ldots, X_{jp}]$ typically correlated. Measurements across trials must be independent.

2) When measurements are observed over time (stock prices or economic indicators), independence may not hold across trials.

Result 3.1 $X_1, \ldots, X_n \equiv$ random sample from joint distribution with mean $\underset{\sim}{\mu}_X = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_p \end{bmatrix}$

and covariance matrix $\Sigma_X = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix}$

$$\bar{X} = \frac{1}{n}\left[X_1 + X_2 + \cdots + X_n\right] = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \vdots \\ \bar{X}_p \end{bmatrix}$$

$$E\{\bar{X}\} = \frac{1}{n}\left[\underset{\sim}{\mu}_X + \underset{\sim}{\mu}_X + \cdots + \underset{\sim}{\mu}_X\right] = \frac{1}{n}\, n\, \underset{\sim}{\mu}_X = \underset{\sim}{\mu}_X$$

~~$Cov\{\bar{X}\} = Cov\left\{\frac{1}{n}\left[X_1 + X_2 + \cdots + X_n\right]\right\}$~~

~~$= \frac{1}{n^2}\left[V\{X_1\} + \cdots + V\{X_n\} + 2\,Cov\{X_1, X_2\} + \cdots + 2\right.$~~

$$Cov\{\bar{X}\} = \frac{1}{n}\Sigma$$

$$E\{S_n\} = \frac{n-1}{n}\Sigma_X = \Sigma - \frac{1}{n}\Sigma$$

$$\Rightarrow E\left\{\frac{n}{n-1} S_n\right\} = \Sigma_X$$

$$\bar{X} - \mu = \frac{1}{n}\left[(\underset{\sim}{X}_1 - \underset{\sim}{\mu}_X) + \cdots + (\underset{\sim}{X}_n - \underset{\sim}{\mu}_X)\right] = \frac{1}{n}\sum_{j=1}^{n}(X_j - \mu_X)$$

$$\Rightarrow (\bar{X} - \mu_X)(\bar{X} - \mu_X)' = \left[\frac{1}{n}\sum_{j=1}^{n}(X_j - \mu_X)\right]\left[\frac{1}{n}\sum_{l=1}^{n}(X_l - \mu_X)\right]'$$

$$= \frac{1}{n^2}\sum_{j=1}^{n}\sum_{l=1}^{n}(\underset{\sim}{X}_j - \underset{\sim}{\mu}_X)(\underset{\sim}{X}_l - \underset{\sim}{\mu}_X)'$$

Note: if $j \neq l$, $E\{(\underset{\sim}{X}_j - \underset{\sim}{\mu}_X)(\underset{\sim}{X}_l - \underset{\sim}{\mu}_X)\} = 0$

(independance across trials)

$$E\{(\underset{\sim}{X}_j - \underset{\sim}{\mu}_X)(\underset{\sim}{X}_j - \underset{\sim}{\mu}_X)'\} = E\left\{\begin{pmatrix} X_{j1} - \mu_1 \\ \vdots \\ X_{jp} - \mu_p \end{pmatrix}\begin{bmatrix} X_{j1} - \mu_1 & \cdots & X_{jp} - \mu_p \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1p} & \sigma_{2p} & \cdots & \sigma_{pp} \end{bmatrix}$$

$$\Rightarrow E\{(\bar{X} - \mu_X)(\bar{X} - \mu_X)'\} = \frac{1}{n^2}\, n\, \Sigma_X = \frac{1}{n}\Sigma_X$$

$$S_n = \frac{1}{n}\sum_{j=1}^{n}(\underset{\sim}{X}_j - \bar{\underset{\sim}{X}})(\underset{\sim}{X}_j - \bar{\underset{\sim}{X}})'$$

$$= \frac{1}{n}\left[\sum_{j=1}^{n}(\underset{\sim}{X}_j - \bar{\underset{\sim}{X}})\underset{\sim}{X}_j' + \sum_{j=1}^{n}(X_j - \bar{X})(-\bar{X})'\right]$$

$$\Rightarrow S_n = \frac{1}{n}\left[\sum_{j=1}^{n} \underset{\sim}{X}_j \underset{\sim}{X}_j' - \bar{\underset{\sim}{X}} \sum_{j=1}^{n} \underset{\sim}{X}_j' - \bar{\underset{\sim}{X}} \sum_{j=1}^{n} (\underset{\sim}{X}_j - \bar{\underset{\sim}{X}})\right]$$

$$= \frac{1}{n}\left[\sum_{j=1}^{n} \underset{\sim}{X}_j \underset{\sim}{X}_j' - n \bar{\underset{\sim}{X}} \bar{\underset{\sim}{X}}'\right]$$

$$\Rightarrow E\{S_n\} = \frac{1}{n} E\left\{\sum_{j=1}^{n} \underset{\sim}{X}_j \underset{\sim}{X}_j' - n \bar{\underset{\sim}{X}} \bar{\underset{\sim}{X}}'\right\}$$

Let V be a vector with mean μ_V, $Cov\{V\} = \Sigma_V$

$$VV' = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_k \end{bmatrix} \begin{bmatrix} V_1 & V_2 & \cdots & V_k \end{bmatrix} = \begin{bmatrix} V_1^2 & V_1V_2 & \cdots & V_1V_k \\ V_1V_2 & V_2^2 & \cdots & V_2V_k \\ \vdots & \vdots & \ddots & \vdots \\ V_1V_k & V_2V_k & \cdots & V_k^2 \end{bmatrix}$$

$$E\{V_1^2\} = V\{V_1\} + [E\{V_1\}]^2 = \Sigma_{V_{11}} + \mu_1^2$$

$$V\{V_1V_2\} = Cov\{V_1, V_2\} + \mu_1\mu_2 = \Sigma_V^{12} + \mu_1\mu_2$$

$$V\{V_iV_k\} = Cov\{V_i, V_j\} + \mu_i\mu_j = \Sigma_V^{ij} + \mu_i\mu_j$$

$$\Rightarrow E\{VV'\} = \Sigma_V + \mu_V\mu_V'$$

$$\Rightarrow E\{\underset{\sim}{X}_j \underset{\sim}{X}_j'\} = \Sigma_X + \underset{\sim}{\mu}_X \underset{\sim}{\mu}_X'$$

$$E\{\bar{\underset{\sim}{X}} \bar{\underset{\sim}{X}}'\} = \frac{1}{n}\Sigma_X + \underset{\sim}{\mu}_X \underset{\sim}{\mu}_X'$$

$$\Rightarrow E\{S_n\} = \frac{1}{n}\left[n\left(\Sigma_x + \underset{\sim}{\mu}_x \underset{\sim}{\mu}_x'\right) - n\left(\frac{1}{n}\Sigma_x + \underset{\sim}{\mu}_x \underset{\sim}{\mu}_x'\right)\right]$$

$$= \frac{1}{n}\left[(n-1)\Sigma_x\right] = \frac{n-1}{n}\Sigma_x$$

$$\Rightarrow E\left\{\frac{n}{n-1} S_n\right\} = E\{S\} = \Sigma_x$$

Unbiased Sample Variance-Covariance Matrix

$$S = \frac{n}{n-1} S_n = \frac{1}{n}\sum_{j=1}^{n}\left(\underset{\sim}{X}_j - \bar{\underset{\sim}{X}}\right)\left(\underset{\sim}{X}_j - \bar{\underset{\sim}{X}}\right)'$$

3.4 Generalized Variance

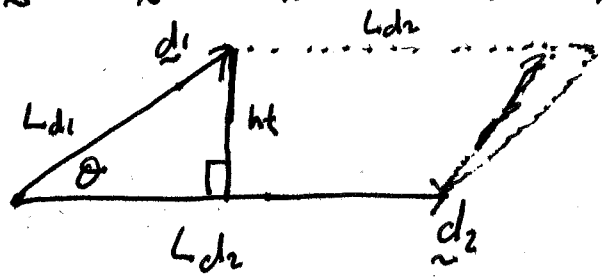
$$S = \begin{bmatrix} s_{11} & \cdots & s_{1p} \\ \vdots & \ddots & \vdots \\ s_{1p} & \cdots & s_{pp} \end{bmatrix} \qquad s_{ik} = \frac{1}{n-1}\sum_{j=1}^{n}\left(x_{ji} - \bar{x}_i\right)\left(x_{jk} - \bar{x}_k\right)$$

S contains p variances and $\binom{p}{2} = \frac{p(p-1)}{2}$ Covariances

Generalized Sample Variance $= |S|$

Geometric Interpretation

$\underset{\sim}{d}_1 = \underset{\sim}{y}_1 - \bar{x}_1 \underset{\sim}{1}$ $\quad$ $\underset{\sim}{d}_2 = \underset{\sim}{y}_2 - \bar{x}_2 \underset{\sim}{1}$



$L_{d_1} = \sqrt{\sum_j (x_{j1} - \bar{x}_1)^2} = \sqrt{(n-1)\, s_{11}}$

$L_{d_2} = \sqrt{\sum_j (x_{j2} - \bar{x}_2)^2} = \sqrt{(n-1)\, s_{22}}$

$\sin\theta = \frac{h_t}{L_{d_1}} \Rightarrow h_t = L_{d_1} \sin\theta$

Area of Trapezoid $= |L_{d_1} \sin\theta| L_{d_2} = L_{d_1} L_{d_2} |\sin\theta|$

$\cos^2\theta + \sin^2\theta = 1 \Rightarrow |\sin\theta| = \sqrt{1 - \cos^2\theta}$

$\Rightarrow$ Area $= L_{d_1} L_{d_2} \sqrt{1 - \cos^2\theta}$ $\quad$ $\cos\theta = r_{12}$

$\Rightarrow$ Area $= (n-1)\sqrt{s_{11}}\sqrt{s_{22}}\sqrt{1 - r_{12}^2}$

$= (n-1)\sqrt{s_{11} s_{22} (1 - r_{12}^2)}$

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} = \begin{pmatrix} s_{11} & r_{12}\sqrt{s_{11}}\sqrt{s_{22}} \\ r_{12}\sqrt{s_{11}}\sqrt{s_{22}} & s_{22} \end{pmatrix}$$

$$|S| = s_{11} s_{22} - \left[r_{12} \sqrt{s_{11}} \sqrt{s_{22}} \right]^2$$

$= s_{11} s_{22} - r_{12}^2 s_{11} s_{22} =$ ~~$s_{11} s_{22} \sqrt{1 - r_{12}^2}$~~

$= s_{11} s_{22} (1 - r_{12}^2) \Rightarrow |S| = \frac{\text{area}^2}{(n-1)^2}$

Generalized to $p-1$ deviation vectors in n space

Generalized Sample variance $= |S| = (n-1)^{-(p-1)}(\text{volume})^2$

p-dimensional space interpretation

$(\underset{\sim}{x} - \bar{\underset{\sim}{x}})' S^{-1} (\underset{\sim}{x} - \bar{\underset{\sim}{x}}) = c^2$ $\Rightarrow$ ellipsis (hyperellipsoid) of point $\underset{\sim}{x}$ a distance of c from $\bar{\underset{\sim}{x}}$.

Volume of $\{x : (\underset{\sim}{x} - \bar{\underset{\sim}{x}})' S^{-1} (\underset{\sim}{x} - \bar{\underset{\sim}{x}}) \leq c^2\} = k_p |S|^{1/2} c^p$

$\Rightarrow$ Volume^2 $= k_p^2 c^{2p} |S|$ = constant $\times$ Generalized sample variance

where: $k_p = \dfrac{2\pi^{p/2}}{p\Gamma(\frac{p}{2})}$

Generalized Variance from Correlation Matrix

Generalized Sample Variance of Standardized vars $= |R|$

Total Sample Variance $= s_{11} + s_{22} + \cdots + s_{pp}$

3.5 Sample Mean, Covariance, Correlation in Matrix Form

$$\underset{\sim}{\bar{X}} = \begin{bmatrix} \bar{X}_1 \\ \vdots \\ \bar{X}_p \end{bmatrix} = \begin{bmatrix} \frac{\underset{\sim}{y}_1' \underset{\sim}{1}}{n} \\ \vdots \\ \frac{\underset{\sim}{y}_p' \underset{\sim}{1}}{n} \end{bmatrix} = \frac{1}{n} X' \underset{\sim}{1}$$

- $n \times p$ matrix of means: $\left[\bar{X}_1 \underset{\sim}{1} ; \bar{X}_2 \underset{\sim}{1} ; \cdots ; \bar{X}_p \underset{\sim}{1} \right]$

$$= \underset{\sim}{1}_n \underset{\sim}{\bar{X}}' = \underset{\sim}{1} \left(\frac{1}{n} X' \underset{\sim}{1} \right)' = \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}' X = \frac{1}{n} J_n X$$

- $n \times p$ matrix of deviations (Residuals):

$$\begin{bmatrix} x_{11} - \bar{x}_1 & \cdots & x_{1p} - \bar{x}_p \\ \vdots & \ddots & \vdots \\ x_{n1} - \bar{x}_1 & \cdots & x_{np} - \bar{x}_p \end{bmatrix} = E$$

- $$X - \frac{1}{n} J_n X = \left(I_n - \frac{1}{n} J_n \right) X \qquad \left(J_n = \underset{\sim}{1}_n \underset{\sim}{1}_n' \right)$$

$p \times p$ matrix of Sums of Squares and cross-products:

$$(n-1)S = \begin{bmatrix} \sum_{j=1}^{n} (x_{j1} - \bar{x}_1)^2 & \cdots & \sum_{j=1}^{n} (x_{j1} - \bar{x}_1)(x_{jp} - \bar{x}_p) \\ \vdots & & \\ \sum_{j=1}^{n} (x_{j1} - \bar{x}_1)(x_{jp} - \bar{x}_p) & \cdots & \sum_{j=1}^{n} (x_{jp} - \bar{x}_p)^2 \end{bmatrix}$$

$$= E'E = \left[\left(I - \frac{1}{n} J \right) X \right]' \left(I - \frac{1}{n} J \right) X = X' \left(I - \frac{1}{n} J \right)' \left(I - \frac{1}{n} J \right) X$$

$(I - \frac{1}{n}J) \equiv$ Symmetric

$(I - \frac{1}{n}J)(I - \frac{1}{n}J) = II - I\frac{1}{n}J - \frac{1}{n}JI + \frac{1}{n}J\frac{1}{n}J$

$= I - \frac{1}{n}J - \frac{1}{n}J + \frac{1}{n}J\frac{1}{n}J$

$\frac{1}{n}J = \frac{1}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \\ 1 & \cdots & 1 \end{bmatrix}$ $\frac{1}{n}J\frac{1}{n}J = \frac{1}{n^2}\begin{bmatrix} n & \cdots & n \\ \vdots & \ddots & \vdots \\ n & \cdots & n \end{bmatrix}$

$= \frac{1}{n}J$

$\Rightarrow (I - \frac{1}{n}J)(I - \frac{1}{n}J) = I - \frac{1}{n}J$ (idempotent)

$\Rightarrow S = \frac{1}{n-1}X'(I - \frac{1}{n}J)X$

Sample Std. Deviation matrix: $D^{1/2} = \begin{bmatrix} \sqrt{s_{11}} & & 0 \\ & \sqrt{s_{22}} & \\ 0 & & \ddots \\ & & & \sqrt{s_{pp}} \end{bmatrix}$

$D^{-1/2} = \begin{bmatrix} 1/\sqrt{s_{11}} & & 0 \\ & 1/s_{22} & \\ 0 & & \ddots \\ & & & 1/\sqrt{s_{pp}} \end{bmatrix}$

$R = D^{-1/2} S D^{-1/2}$

$S = D^{1/2} R D^{1/2}$

3.6. Sample Values of Linear Combination of Vars

$$\underset{\sim}{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} \qquad \underset{\sim}{c}'\underset{\sim}{X} = c_1 X_1 + \cdots + c_p X_p \quad \text{(General form)}$$

observed value on j^{th} trial:

$$\underset{\sim}{c}'\underset{\sim}{x}_j = c_1 x_{j1} + \cdots + c_p x_{jp} \qquad j = 1, \ldots, n$$

Sample mean: $\dfrac{\underset{\sim}{c}'\underset{\sim}{x}_1 + \cdots + \underset{\sim}{c}'\underset{\sim}{x}_n}{n} = \dfrac{1}{n}\underset{\sim}{c}'(\underset{\sim}{x}_1 + \ldots + \underset{\sim}{x}_n)$

$$= \underset{\sim}{c}'\bar{\underset{\sim}{x}}$$

Sample variance: $\dfrac{1}{n-1}\sum_{j=1}^{n}\left(\underset{\sim}{c}'\underset{\sim}{x}_j - \underset{\sim}{c}'\bar{\underset{\sim}{x}}\right)^2$

$$= \frac{1}{n-1}\sum_{j=1}^{n}\left[c'(\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})\right]^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left[\underset{\sim}{c}'(\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})(\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})'\underset{\sim}{c}\right]$$

$$= \underset{\sim}{c}'\,\frac{\sum_{j=1}^{n}(\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})(\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})'}{n-1}\,\underset{\sim}{c} = \underset{\sim}{c}' S \underset{\sim}{c}$$

Second Linear combination: $b'X$

Sample mean: $\underset{\sim}{b}'\bar{\underset{\sim}{x}}$ Sample variance: $\underset{\sim}{b}' S \underset{\sim}{b}$

Sample Covariance

$$\frac{(\underset{\sim}{b}_1'\underset{\sim}{x}_1 - \underset{\sim}{b}_1'\bar{\underset{\sim}{x}})(\underset{\sim}{c}_1'\underset{\sim}{x}_1 - \underset{\sim}{c}_1'\bar{\underset{\sim}{x}}) + \cdots + (\underset{\sim}{b}_1'\underset{\sim}{x}_n - \underset{\sim}{b}_1'\bar{\underset{\sim}{x}})(\underset{\sim}{c}_1'\underset{\sim}{x}_n - \underset{\sim}{c}_1'\bar{\underset{\sim}{x}})}{n-1}$$

$$= \frac{b'(\underset{\sim}{x}_1 - \bar{\underset{\sim}{x}})(\underset{\sim}{x}_1 - \bar{\underset{\sim}{x}})'c + \cdots + b'(\underset{\sim}{x}_n - \bar{\underset{\sim}{x}})(\underset{\sim}{x}_n - \bar{\underset{\sim}{x}})'c}{n-1}$$

$$= b'Sc = Cov\{b'X, c'X\}$$

Generalized to q linear combinations:

$$a_{i1}X_1 + a_{i2}X_2 + \cdots + a_{ip}X_p \qquad i = 1, \ldots, q$$

$$\Rightarrow AX \quad w/ \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{q1} & \cdots & a_{qp} \end{bmatrix}$$

Sample Mean: $A\bar{\underset{\sim}{x}}$ Sample covariance matrix: ASA'