Chapter 2

2.1.6 Length $\equiv L_x = \sqrt{\underset{\sim}{x}'\underset{\sim}{x}} = \sqrt{5^2+1^2+3^2} = \sqrt{35} = 5.9161$

$$\cos\theta = \frac{x'y}{\sqrt{x'x}\sqrt{y'y}} \qquad x'y = 5(-1)+1(3)+3(1) = 1$$
$$y'y = (-1)^2+3^2+1^2 = 11$$

$$\cos\theta = \frac{1}{\sqrt{35}\sqrt{11}} = \frac{1}{19.6214} = .0510$$

$$\Rightarrow \theta = \cos^{-1}(.0510) = 87.0787$$

Projection of $\underset{\sim}{y}$ on $\underset{\sim}{x}$: $\frac{x'y}{x'x}\underset{\sim}{x} =$

$$= \frac{1}{35}\begin{bmatrix}5\\1\\3\end{bmatrix} = \begin{bmatrix}.1429\\.0286\\.0857\end{bmatrix}$$

2.2 $A = \begin{bmatrix}-1 & 3\\4 & 2\end{bmatrix}$ $B = \begin{bmatrix}4 & -3\\1 & -2\\-2 & 0\end{bmatrix}$ $C = \begin{bmatrix}5\\-4\\2\end{bmatrix}$

a) $5A = \begin{bmatrix}-5 & 15\\20 & 10\end{bmatrix}$ b) $BA = \begin{bmatrix}-6 & 6\\-9 & -1\\2 & -6\end{bmatrix}$

c) $A'B' = (BA)' = \begin{bmatrix}-6 & -9 & 2\\6 & -1 & -6\end{bmatrix}$

d) $C'B = \begin{bmatrix}12 & -7\end{bmatrix}$

e) AB Does not exist $c_A \neq r_B$

2.3 $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 0 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

a) $A' = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ $(A')' = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = A$

b) $C^{-1} = \frac{1}{2-12} \begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0.4 \\ 0.3 & -0.1 \end{bmatrix}$

$C' = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ $(C')^{-1} = \frac{1}{2-12} \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$= \begin{bmatrix} -0.2 & 0.3 \\ 0.4 & -0.1 \end{bmatrix} = (C^{-1})'$

c) $AB = \begin{bmatrix} 7 & 8 & 7 \\ 16 & 4 & 11 \end{bmatrix}$ $(AB)' = \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix}$

$B'A' = \begin{bmatrix} 1 & 5 \\ 4 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix} = (AB)'$

d) $\cancel{AB_{ij} = \sum_{l=1}^{k} (A_{il} + B_{lj})}$ $AB_{11} = \sum_{l=1}^{k} A_{1l} B_{l1}$

$\cancel{B'A'_{ij} = \sum_{l=1}^{k} (B_{lj} + A_{il})}$ $AB_{ij} = \sum_{l=1}^{k} A_{il} B_{lj}$ $i = 1, \ldots, r_A$, $j = 1, \ldots, c_B$

$B'A'_{ij} = \sum_{l=1}^{k} B_{li} A_{jl}$ $i = 1, \ldots, r_A$, $j = 1, \ldots, c_B$

2.4 A^{-1}, B^{-1} exist

a) $(A')^{-1} = (A^{-1})'$

$$AA^{-1} = I = I' = (AA^{-1})' = (A^{-1})'A' = I$$

$$(A')^{-1}A' = I \Rightarrow (A^{-1})' = (A')^{-1}$$

b) $(AB)^{-1} = B^{-1}A^{-1}$

$$(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$$

$$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$$

2.5 $Q = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ -\frac{12}{13} & \frac{5}{13} \end{bmatrix}$ $\quad Q' = \begin{bmatrix} 5/13 & -12/13 \\ 12/13 & 5/13 \end{bmatrix}$

$$QQ' = \begin{bmatrix} \left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 & \frac{5}{13}\left(-\frac{12}{13}\right) + \frac{12}{13}\left(\frac{5}{13}\right) \\ -\frac{12}{13}\left(\frac{5}{13}\right) + \frac{5}{13}\left(\frac{12}{13}\right) & \left(-\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{25+144}{169} & 0 \\ 0 & \frac{144+25}{169} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

2.6 $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$ a) $A = A' \Rightarrow$ Symmetric

$$\underline{x}'A\underline{x} = 9x_1^2 + 6x_2^2 - 2(2)x_1x_2$$

$$= 5x_1^2 + 5x_2^2 + 4x_1^2 + x_2^2 - 2(2)x_1x_2$$

$$= 5(x_1^2 + x_2^2) + (2x_1 - x_2)^2$$

$\Rightarrow x'Ax > 0$ unless $x_1 = x_2 = 0 \Rightarrow A \equiv$ p.d.

2.7. a) $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{vmatrix} = 0$

$$= (9-\lambda)(6-\lambda) - (-2)(-2) = 54 - 15\lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 15\lambda + 50 = 0 \Rightarrow \lambda = \frac{-(-15) \pm \sqrt{(-15)^2 - 4(1)(50)}}{2(1)}$$

$$= \frac{15 \pm \sqrt{225 - 200}}{2} = \frac{15 \pm 5}{2} \equiv \overset{=\lambda_1}{10},\ 5 = \lambda_2$$

$$A\underset{\sim}{x_1} = \lambda_1 \underset{\sim}{x_1} \Rightarrow \begin{matrix} 9x_1 - 2x_2 = 10x_1 \\ -2x_1 + 6x_2 = 10x_2 \end{matrix} \Rightarrow x_1 = -2x_2$$

$$\Rightarrow \underset{\sim}{x_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \underset{\sim}{x_1}'\underset{\sim}{x_1} = 5 \Rightarrow \underset{\sim}{e_1} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

2.7a) continued

$$A\underset{\sim}{x_2} = \lambda_2 \underset{\sim}{x_2} \Rightarrow \begin{matrix} 9x_1 - 2x_2 = 5x_1 \\ -2x_1 + 6x_2 = 5x_2 \end{matrix} \Rightarrow x_2 = 2x_1$$

$$\Rightarrow \underset{\sim}{x_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \underset{\sim}{x_2}'\underset{\sim}{x_2} = 5 \Rightarrow \underset{\sim}{e_2} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

b) $A = \lambda_1 \underset{\sim}{e_1}\underset{\sim}{e_1}' + \lambda_2 \underset{\sim}{e_2}\underset{\sim}{e_2}'$

$$\underset{\sim}{e_1}\underset{\sim}{e_1}' = \begin{bmatrix} 4/5 & -\frac{2}{5} \\ -2/5 & 1/5 \end{bmatrix} \Rightarrow \lambda_1 \underset{\sim}{e_1}\underset{\sim}{e_1}' = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix}$$

$$\underset{\sim}{e_2}\underset{\sim}{e_2}' = \begin{bmatrix} \frac{1}{5} & 2/5 \\ 2/5 & 4/5 \end{bmatrix} \Rightarrow \lambda_2 \underset{\sim}{e_2}\underset{\sim}{e_2}' = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\lambda_1 \underset{\sim}{e_1}\underset{\sim}{e_1}' + \lambda_2 \underset{\sim}{e_2}\underset{\sim}{e_2}' = \begin{bmatrix} 8+1 & -4+2 \\ -4+2 & 2+4 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} = A$$

c) $A^{-1} = \frac{1}{\lambda_1} \underset{\sim}{e_1}\underset{\sim}{e_1}' + \frac{1}{\lambda_2} \underset{\sim}{e_2}\underset{\sim}{e_2}'$

$$= \begin{bmatrix} \frac{4}{50} & \frac{-2}{50} \\ -\frac{2}{50} & \frac{1}{50} \end{bmatrix} + \begin{bmatrix} \frac{2}{50} & \frac{4}{50} \\ \frac{4}{50} & \frac{8}{50} \end{bmatrix} = \begin{bmatrix} \frac{6}{50} & \frac{2}{50} \\ \frac{2}{50} & \frac{9}{50} \end{bmatrix}$$

$$|A^{-1} - \lambda I| = \begin{vmatrix} \frac{6}{50} - \lambda & +\frac{2}{50} \\ +\frac{2}{50} & \frac{9}{50} - \lambda \end{vmatrix} \cancel{= \left(\frac{4}{50} - \lambda\right)\left(\frac{1}{50} - \lambda\right) - 2\left(\frac{2}{50}\right) = 0}$$

2.7.c continued

$$\left(\frac{6}{50}-\lambda\right)\left(\frac{9}{50}-\lambda\right)-\left(\frac{2}{50}\right)\left(\frac{2}{50}\right)=\frac{54}{2500}-\frac{15}{50}\lambda+\lambda^2-\frac{4}{50}=0$$

$$\Rightarrow \lambda^2-\frac{3}{10}\lambda+\frac{1}{50}=0$$

$$\Rightarrow \lambda=\frac{-\left(-\frac{3}{10}\right)\pm\sqrt{\left(-\frac{3}{10}\right)^2-4(1)\left(\frac{1}{50}\right)}}{2(1)}$$

$$=\frac{\frac{3}{10}\pm\sqrt{\frac{9}{100}-\frac{8}{100}}}{2}=\frac{\frac{3}{10}\pm\frac{1}{10}}{2}$$

$$=\frac{\frac{4}{10}}{2},\ \frac{\frac{2}{10}}{2}\Rightarrow \lambda_1=0.20,\quad \lambda_2=0.10$$

$$A^{-1}\underset{\sim}{x}_1=\lambda_1 x_1 \Rightarrow \begin{aligned}\frac{6}{50}x_1+\frac{2}{50}x_2&=\frac{10}{50}x_1\\ \frac{2}{50}x_1+\frac{9}{50}x_2&=\frac{10}{50}x_2\end{aligned}$$

$$\Rightarrow x_2=2x_1 \Rightarrow \underset{\sim}{x}_1=\begin{bmatrix}1\\2\end{bmatrix}\Rightarrow \underset{\sim}{e}_1=\begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$$

$$A^{-1}\underset{\sim}{x}_2=\lambda_2\underset{\sim}{x}_2 \Rightarrow \begin{aligned}\frac{6}{50}x_1+\frac{2}{50}x_2&=\frac{5}{50}x_1\\ \frac{2}{50}x_1+\frac{9}{50}x_2&=\frac{5}{50}x_2\end{aligned}$$

$$\Rightarrow x_1=-2x_2 \Rightarrow \underset{\sim}{x}_2=\begin{bmatrix}-2\\1\end{bmatrix}\Rightarrow \underset{\sim}{e}_2=\begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$$

2.8 $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$

$$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) - 2(2)$$

$$= -2 - \lambda + 2\lambda + \lambda^2 - 4 = \lambda^2 + \lambda - 6 = 0$$

Quadratic:
$$\lambda = \pm \frac{-1 \pm \sqrt{1^2 - 4(1)(-6)}}{2(1)} = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2}$$

$$= 2, -3$$

$$A\underset{\sim}{x}_1 = \lambda_1 \underset{\sim}{x}_1 \Rightarrow \begin{pmatrix} x_1 + 2x_2 = 2x_1 \\ 2x_1 - 2x_2 = 2x_2 \end{pmatrix} \Rightarrow x_1 = 2x_2$$

$$\underset{\sim}{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \underset{\sim}{e}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$A\underset{\sim}{x}_2 = \lambda_2 \underset{\sim}{x}_2 \Rightarrow \begin{matrix} x_1 + 2x_2 = -3x_1 \\ 2x_1 - 2x_2 = -3x_2 \end{matrix} \Rightarrow x_2 = -2x_1$$

$$\underset{\sim}{x}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow \underset{\sim}{e}_2 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

$$A = 2\begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} + (-3)\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{8-3}{5} & \frac{4+6}{5} \\ \frac{4+6}{5} & \frac{2-12}{5} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = A$$

2.9.

$$A^{-1} = \frac{1}{2}\begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} + \left(-\frac{1}{3}\right)\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{10} \end{bmatrix} + \begin{bmatrix} -\frac{1}{15} & \frac{2}{15} \\ \frac{2}{15} & -\frac{4}{15} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12-2}{30} & \frac{6+4}{30} \\ \frac{6+4}{30} & \frac{3-8}{30} \end{bmatrix} = \begin{bmatrix} \frac{10}{30} & \frac{10}{30} \\ \frac{10}{30} & -\frac{5}{30} \end{bmatrix} = \begin{bmatrix} \frac{2}{6} & \frac{2}{6} \\ \frac{2}{6} & -\frac{1}{6} \end{bmatrix}$$

Check $AA^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} \frac{2}{6} & \frac{2}{6} \\ \frac{2}{6} & -\frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2.11 Definition 2.A.24: $|A| = a_{11}$ if $k=1$

$$|A| = \sum_{j=1}^{k} a_{1j}\,|A_{1j}|\,(-1)^{1+j} \quad k>1$$

$A_{1j} \equiv$ matrix formed by deleting row 1, col j of A.

$A \equiv$ diagonal $p \times p$ matrix $\Rightarrow A = \begin{bmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & \ddots \; a_{pp} \end{bmatrix}$

2.11 continued

$$|A| = a_{11}|A_{11}|(-1)^{1+1} + a_{12}|A_{12}|(-1)^{1+2} + \dots + a_{1p}|A_{1p}|(-1)^{1+p}$$

where $a_{12} = \dots = a_{1p} = 0$

$$\Rightarrow |A| = a_{11}|A_{11}|$$

$$A_{11} = \begin{bmatrix} a_{22} & & 0 \\ & \ddots & \\ 0 & & a_{pp} \end{bmatrix} \Rightarrow |A_{11}| = a_{22}|A_{12,12}|(-1)^{1+1} + (p-2)(0)$$

$$= |A| = a_{11}a_{22}|A_{12,12}|$$

$$A_{12,12} = \begin{bmatrix} a_{33} & & 0 \\ & \ddots & \\ 0 & & a_{pp} \end{bmatrix}$$

keeps going until

$$A_{12\dots p-1,\,12\dots p-1} = [a_{pp}]$$

$$\Rightarrow |A| = \prod_{i=1}^{p} a_{ii}$$

2.12 A = $p \times p$ symmetric matrix $\Rightarrow A = \sum_{i=1}^{p} \lambda_i \underset{\sim}{e}_i \underset{\sim}{e}_i' = P \Lambda P'$

$$P = [\underset{\sim}{e}_1 \vdots \underset{\sim}{e}_2 \vdots \cdots \vdots \underset{\sim}{e}_p] \qquad P'P = \begin{bmatrix} e_1'e_1 & e_1'e_2 & \dots & e_1'e_p \\ e_2'e_1 & e_2'e_2 & \dots & e_2'e_p \\ \vdots & \vdots & \ddots & \\ e_p'e_1 & e_p'e_2 & \dots & e_p'e_p \end{bmatrix}$$

$$I =$$

Result 2A.11(e): $|AB| = |A||B|$

also $|A| = |A'|$ (2A.11 a)

2.12 continued

$$|A| = |P\Lambda P'| = |P||\Lambda P'| = |P||\Lambda||P'|$$

$$= |P'||P||\Lambda| = |P'P||\Lambda| = |I||\Lambda| = |\Lambda| = \prod_{i=1}^{p} \lambda_i$$

2.13

$Q \equiv p \times p$ orthogonal matrix $\Rightarrow QQ' = I$

2.A.4.a $|A| = |A'| \Rightarrow |Q| = |Q'| \Rightarrow |QQ'| = |Q|^2$

$$|QQ'| = |I| = 1 = |Q|^2 \Rightarrow |Q| = \pm 1.$$

2.15

$$\underset{\sim}{x}' A \underset{\sim}{x} = 3x_1^2 + 3x_2^2 - 2x_1x_2$$

$$= 2x_1^2 + 2x_2^2 + (x_1^2 - 2x_1x_2 + x_2^2)$$

$$= 2(x_1^2 + x_2^2) + (x_1 - x_2)^2 \geq 0$$

$$= 0 \quad \text{iff} \quad x_1 = x_2 = 0$$

2.16

$$A \equiv n \times p = \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ a_{n1} & \cdots & a_{np} \end{bmatrix}$$

$$A'A = \begin{bmatrix} \sum_{j=1}^{n} a_{j1}^2 & \sum_{j=1}^{n} a_{j1}a_{j2} & \cdots & \sum_{j=1}^{n} a_{j1}a_{jp} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} a_{jp}a_{j1} & \sum_{j=1}^{n} a_{jp}a_{j2} & \cdots & \sum_{j=1}^{n} a_{jp}^2 \end{bmatrix} \equiv \text{Symmetric}$$

2.19 $\underset{m\times m}{A^{1/2}} = \sum_{i=1}^{m} \sqrt{\lambda_i}\, \underset{\sim}{e_i}\, \underset{\sim}{e_i}' = P\Lambda^{1/2}P' \quad PP' = P'P = I$

1) $(A^{1/2})' = (P\Lambda^{1/2}P')' = P(\Lambda^{1/2})'P' = P\Lambda^{1/2}P' = A^{1/2}$

2) $A^{1/2}A^{1/2} = P\Lambda^{1/2}P'P\Lambda^{1/2}P' = P\Lambda^{1/2}I\Lambda^{1/2}P'$

$= P\Lambda^{1/2}\Lambda^{1/2}P' = P\Lambda P' = A$

3) $(A^{1/2})^{-1}A^{1/2} = I \qquad (A^{1/2})^{-1} = \sum_{i=1}^{m} \frac{1}{\sqrt{\lambda_i}} e_i e_i' = P\Lambda^{-1/2}P'$

$P\Lambda^{-1/2}P'P\Lambda^{1/2}P' = P\Lambda^{-1/2}\Lambda^{1/2}P' = PIP' = PP' = I$

4) $A^{1/2}A^{-1/2} = I$ (clearly from above) $\Lambda^{1/2}\Lambda^{-1/2} = \Lambda^{-1/2}\Lambda^{1/2} = I$

$A^{-1/2}A^{-1/2} = P\Lambda^{-1/2}P'P\Lambda^{-1/2}P' = P\Lambda^{-1/2}\Lambda^{-1/2}P' = P\Lambda^{-1}P' = A^{-1}$

2.20 Exercise 2.3 $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

$0 = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - (1)(1)$

$= 6 - 5\lambda + \lambda^2 - 1 = \lambda^2 - 5\lambda + 5 = 0$

$\Rightarrow \lambda = \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(5)}}{2(1)} = \dfrac{5 \pm \sqrt{5}}{2}$

$= \left(\dfrac{7.2361}{2}, \dfrac{2.7639}{2}\right) = (3.6181, 1.3820)$

2.20 continued

$$A\underset{\sim}{x}_1 = \lambda_1 x_1 \Rightarrow \begin{matrix} 2x_1 + x_2 = 3.6181 x_1 \\ x_1 + 3x_2 = 3.6181 x_2 \end{matrix} \Rightarrow x_2 = 1.6181\, x_1$$

$$\Rightarrow \underset{\sim}{x}_1 = \begin{bmatrix} 1 \\ 1.6181 \end{bmatrix} \Rightarrow \underset{\sim}{e}_1 = \begin{bmatrix} 1/1.9022 \\ 1.6181/1.9022 \end{bmatrix} = \begin{bmatrix} 0.5257 \\ 0.8506 \end{bmatrix}$$

$$A\underset{\sim}{x}_2 = \lambda_2 \underset{\sim}{x}_2 \Rightarrow \begin{matrix} 2x_1 + x_2 = 1.3820 x_1 \\ x_1 + 3x_2 = 1.3820 x_2 \end{matrix} \Rightarrow x_2 = -.6180\, x_1$$

$$\Rightarrow \underset{\sim}{x}_2 = \begin{bmatrix} 1 \\ -.6180 \end{bmatrix} \Rightarrow e_1 = \begin{bmatrix} 1/1.1756 \\ -.6180/1.1756 \end{bmatrix} = \begin{bmatrix} .8506 \\ -.5257 \end{bmatrix}$$

$$A^{1/2} = \sqrt{3.6181}\begin{bmatrix} .2764 & .4472 \\ .4472 & .7235 \end{bmatrix} + \sqrt{1.3820}\begin{bmatrix} .7235 & -.4472 \\ -.4472 & .2764 \end{bmatrix}$$

$$\sqrt{3.6181} = 1.9021 \qquad \sqrt{1.3820} = 1.1756$$

$$\frac{1}{\sqrt{3.6181}} = .5257 \qquad \frac{1}{\sqrt{1.3820}} = .8506$$

$$\Rightarrow A^{1/2} = \begin{bmatrix} 1.9021(.2764) + 1.1756(.7235) & 1.9021(.4472) + 1.1756(-.4472) \\ 1.9021(.4472) + 1.1756(-.4472) & 1.9021(.7235) + 1.1756(.2764) \end{bmatrix}$$

$$= \begin{bmatrix} 1.3763 & .3249 \\ .3249 & 1.7011 \end{bmatrix}$$

2.20 continued

$$A^{-1/2} = \begin{bmatrix} .5257(.2764)+.8506(.7235) & .5257(.4472)+.8506(-.4472) \\ .5257(.4472)+.8506(-.4472) & .5257(.7235)+.8506(.2764) \end{bmatrix}$$

$$= \begin{pmatrix} .7607 & -.1453 \\ -.1453 & .6154 \end{pmatrix}$$

$$A^{1/2}A^{-1/2} = \begin{bmatrix} 1.3763(.7607)+(.3249)(-.1453) & 1.3763(-.1453)+.3249(\cancel{}\,.6154) \\ .3249(.7607)+(1.7011)(-.1453) & .3249(-0.1453)+1.7011(.6154) \end{bmatrix}$$

$$= \begin{bmatrix} 0.9997 & .0000 \\ 0.0000 & 0.9996 \end{bmatrix} \approx I \text{ (Round-off)}$$

2.23

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}} \qquad \sigma_{ij} = \rho_{ij}\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}$$

$$\rho = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{12} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & & \\ \rho_{1p} & \rho_{2p} & \cdots & 1 \end{bmatrix} \qquad V^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & & 0 \\ & \sqrt{\sigma_{22}} & \\ 0 & & \ddots \\ & & & \sqrt{\sigma_{pp}} \end{bmatrix}$$

$$V^{1/2}\rho = \begin{bmatrix} \sqrt{\sigma_{11}} & \sqrt{\sigma_{11}}\,\rho_{12} & \cdots & \sqrt{\sigma_{11}}\,\rho_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\sigma_{pp}}\,\rho_{1p} & \sqrt{\sigma_{pp}}\,\rho_{2p} & \cdots & \sqrt{\sigma_{pp}} \end{bmatrix}$$

2.23 Continued

2.17

$$\Rightarrow V^{1/2}\rho V^{1/2} = \begin{bmatrix} \sigma_{11} & \rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}} & \cdots & \rho_{1p}\sqrt{\sigma_{11}}\sqrt{\sigma_{pp}} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\sigma_{11}}\sqrt{\sigma_{pp}}\rho_{1p} & \sqrt{\sigma_{22}}\sqrt{\sigma_{pp}}\rho_{2p} & \cdots & \sigma_{pp} \end{bmatrix} = \Sigma$$

$$\Rightarrow V^{-1/2}\Sigma V^{-1/2} = V^{-1/2}V^{1/2}\rho V^{1/2}V^{-1/2} = \rho$$

2.24 $\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

a) $\Sigma^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) $|\Sigma - \lambda I| = (4-\lambda)(9-\lambda)(1-\lambda) = 0$

$\Rightarrow \lambda_1 = 9, \quad \lambda_2 = 4, \quad \lambda_3 = 1$

c) $|\Sigma^{-1} - \lambda I| = 0 \Rightarrow \lambda_1 = 1, \quad \lambda_2 = \frac{1}{4}, \quad \lambda_3 = \frac{1}{9}$

2.25 $\Sigma = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$

a) $\rho = \begin{bmatrix} 1 & \frac{-2}{5(2)} & \frac{4}{5(3)} \\ -\frac{2}{5(2)} & 1 & \frac{1}{2(3)} \\ \frac{4}{5(3)} & \frac{1}{2(3)} & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{5} & \frac{4}{15} \\ -\frac{1}{5} & 1 & \frac{1}{6} \\ \frac{4}{15} & \frac{1}{6} & 1 \end{bmatrix}$

$V^{1/2} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

b) $V^{1/2}\rho = \begin{bmatrix} 5 & -1 & \frac{4}{3} \\ -\frac{2}{5} & 2 & \frac{1}{3} \\ \frac{4}{5} & \frac{1}{2} & 3 \end{bmatrix} \Rightarrow V^{1/2}\rho V^{1/2} = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} = \Sigma$

2.26 a) $\rho_{13} = \frac{4}{\sqrt{25}\sqrt{9}} = \frac{4}{15}$

b) $Corr\{X_1, \frac{1}{2}X_2 + \frac{1}{2}X_3\} = ?$

$Cov\{X_1, \frac{1}{2}X_2 + \frac{1}{2}X_3\} = \frac{1}{2}Cov\{X_1, X_2\} + \frac{1}{2}Cov\{X_1, X_3\}$

$= \frac{1}{2}[-2 + 4] = 1$

$V\{X_1\} = 25 \quad V\{\frac{1}{2}(X_2 + X_3)\} = \frac{1}{4}[4 + 9 + 2(1)] = \frac{15}{4}$

2.26 b Continued

$$Corr\{X_1, \tfrac{1}{2}X_1 + \tfrac{1}{2}X_3\} = \frac{1}{\sqrt{25}\sqrt{\frac{15}{4}}} = \frac{1}{2.5\sqrt{15}} = .1033$$

2.27 $E\{\underline{X}\} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$ $V\{X\} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$

a) $E\{X_1 - 2X_2\} = \mu_1 - 2\mu_2$

$V\{X_1 - 2X_2\} = \sigma_{11} + 4\sigma_{22} + 2(1)(-2)\sigma_{12} = \sigma_{11} + 4\sigma_{22} - 4\sigma_{12}$

b) $E\{-X_1 + 3X_2\} = -\mu_1 + 3\mu_2$

$V\{-X_1 + 3X_2\} = \sigma_{11} + 9\sigma_{22} - 6\sigma_{12}$

c) $E\{X_1 + X_2 + X_3\} = \mu_1 + \mu_2 + \mu_3$

$V\{X_1 + X_2 + X_3\} = \sigma_{11} + \sigma_{22} + \sigma_{33} + 2\sigma_{12} + 2\sigma_{13} + 2\sigma_{23}$

e) $E\{X_1 + 2X_2 - X_3\} = \mu_1 + 2\mu_2 - \mu_3$

$V\{X_1 + 2X_2 - X_3\} = \sigma_{11} + 4\sigma_{22} + \sigma_{33} + 4\sigma_{12} - 2\sigma_{13} - 4\sigma_{23}$

f) $3X_1 - 4X_2$ X_1, X_2 independent

$E\{3X_1 - 4X_2\} = 3\mu_1 - 4\mu_2$

$V\{3X_1 - 4X_2\} = 9\sigma_{11} + 16\sigma_{22}$