STA 6246 - Homework No. 2, Fall, 2003
Due Wednesday, September 24
Do all problems, but turn in only 1, 2, and 5

1. **Problem 1**

   Suppose that $\mathbf{x}$ is distributed as $N(\mu, \mathbf{\Sigma})$ and that $\mathbf{A\Sigma A} = \mathbf{A}$

   (a) Show that $(\mathbf{x} - \mathbf{c})'\mathbf{A}(\mathbf{x} - \mathbf{c}) \sim \chi'^2_p(\lambda)$, where $p = rank(\mathbf{A})$. What is λ?

   (b) Show that $\mathbf{A} \geq 0$, i.e., $\mathbf{A}$ is positive semidefinite.

2. **Problem 2**

   Let $\mathbf{x} = (x_1, x_2)' \sim N(\mu, \mathbf{\Sigma})$, where

   $$\mathbf{\Sigma} = (1 - \rho)\mathbf{I}_2 + \rho\mathbf{J}_2 \tag{1}$$

   Let $Q_1 = (x_1 - x_2)^2$ and $Q_2 = (x_1 + x_2)^2$. Show that $Q_1/[2(1 - \rho)]$ has a chi-squared distribution and that Q_1 and Q_2 are distributed independently.

3. **Problem 3**

   Let $\mathbf{y} = (y_1, y_2, y_3)' \sim N(\mu, \mathbf{\Sigma})$, where $\mu = (0, 1, 2)'$, and

   $$\mathbf{\Sigma} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 9 \end{bmatrix} \tag{2}$$

   Let $z = (2, -1, -1)\mathbf{y}$. Find a random variable $u = \mathbf{b}'\mathbf{y}$, where $\mathbf{b} = (b_1, b_2, b_3)'$ such that $E(u) = 0$ and u is independent of z.

4. **Problem 4**

   Let $\mathbf{y}_1 = (y_{11}, y_{12}, \cdots, y_{1n_1})'$, and $\mathbf{y}_2 = (y_{21}, y_{22}, \cdots, y_{2n_2})'$ be samples drawn from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively, and the samples are independent.

   Find the distribution of

   $$V = \frac{(\bar{y}_{1.} - \bar{y}_{2.})^2/(\frac{1}{n_1} + \frac{1}{n_2})}{\sum_{i=1}^{2}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_{i.})^2/(n_1 + n_2 - 2)} \tag{3}$$

   when $\sigma_1^2 = \sigma_2^2$.

5. **Problem 5**

   Consider the balanced random one-way model,

   $$y_{ij} = \mu + \alpha_i + \epsilon_{ij}, \tag{4}$$

   $i = 1, 2, \cdots, a$, $j = 1, 2, \cdots, n$, $\alpha_i \sim N(0, \sigma_\alpha^2)$, $\epsilon_{ij} \sim N(0, \sigma_\epsilon^2)$, the α_i's are independent, the ϵ_{ij}'s are independent, and the α_i's are independent of the ϵ_{ij}'s.

(a) Write the model in vector form using direct products.

(b) Give $\boldsymbol{\Sigma}$, the variance-covariance matrix of $\mathbf{y}$, the vector of observations, using direct products.

(c) What is the distribution of SS_A, the sum of squartes associated with the α_i's?