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# Comparison of designs for the three-fold nested random model

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The quality of estimation of variance components depends on the design used as well as on the unknown values of the variance components. In this article, three designs are compared, namely, the balanced, staggered, and inverted nested designs for the three-fold nested random model. The comparison is based on the so-called quantile dispersion graphs using analysis of variance (ANOVA) and maximum likelihood (ML) estimates of the variance components. It is demonstrated that the staggered nested design gives more stable estimates of the variance component for the highest nesting factor than the balanced design. The reverse, however, is true in case of lower nested factors. A comparison between ANOVA and ML estimation of the variance components is also made using each of the aforementioned designs.

**Keywords:** ANOVA; balanced nested design; four-stage nested experiment; inverted nested design; maximum likelihood; quantile dispersion graphs; staggered nested design; variance components

## 1. Introduction

Nested designs are used in experiments consisting of several stages that can be carried out in a hierarchical order. Each stage is identified with a particular factor that affects a response of interest. These designs are applicable in a wide variety of experimental situations such as in, for example, agricultural and biological researches, geology [1,11,24], pharmaceutics [17,28], and the chemical industry [10,26,29], to name just a few [9]. A nested design is said to be *balanced* if the number of levels of each factor is the same for all levels of the factor preceding it in the hierarchical order of the design.

A balanced nested design is easy to administer and its statistical analysis is simple. Its main disadvantage, however, is its uneven allocation of the degrees of freedom to the various factors (or stages). In particular, the degrees of freedom are too heavily concentrated in lower nested factors. Hence, more information can be available regarding the lower stages than the higher ones. Getting more degrees of freedom for the latter would therefore require a considerable increase in the size of the experiment. If all factors are random, then the quality of estimation of the corresponding variance components can be heavily influenced by the choice of design. With a balanced design, the variance components can be estimated less precisely in the higher nested stages than in the

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lower ones. To remedy this situation, the degrees of freedom for the various stages have to be allocated differently using particular types of unbalanced nested designs.

Bainbridge [2,3] introduced the so-called *staggered nested design* (SND) and the *inverted nested design* (IND) for a four-stage nested experiment. The SND is a partially replicated nested design with about the same degrees of freedom for each stage. More specifically, the number of degrees of freedom for the highest(top) stage is equal to $a$–1, where $a$ is the number of levels of the factor used in that stage. All lower stages have $a$ degrees of freedom each. The IND is a highly fractionated nested design, and the degrees of freedom are concentrated near the top of the design.

Leone and Nelson [15] investigated the empirical distribution of the analysis of variance (ANOVA) estimators of the variance components for the four-stage balanced nested design using several sets of variance components. Leone et al. [16] discussed and compared the SND and the IND. The comparison was based on the probability of a negative ANOVA estimator of a variance component. They also plotted the empirical distributions of such an estimator for both designs. It was concluded that no single design was best for estimating all the variance components. The SND was recommended as a 'good compromise choice', not only because it spreads the degrees of freedom almost evenly over all stages, but also because it is simple to administer due to its 'open-ended' structure, which means that 'the design allows more levels of the top-stage factor to be added without upsetting the analysis' ([16], p. 737). More recently, Ojima [23] proposed a class of unbalanced nested designs that was a generalization of the SND. Some of these designs were found to be more efficient than the traditional SND in estimating some of the variance components.

The estimation problem of the variance components for the SND has also been addressed by several other authors, including Smith and Beverly [27], Nelson [19–21], Khattree et al. [7], and Ojima [22]. In addition, Khattree and Naik [6] presented statistical tests for the variance components associated with the SND.

It should be pointed out that the choice of design for estimating variance components depends not only on the design criterion, but also on the true values of the variance components. Although the SND is regarded as the most popular unbalanced nested design [20], comparisons of designs were previously made using some prior knowledge of the true values of the variance components. It would therefore be desirable to compare designs without having to specify such values. The so-called quantile dispersion graphs (QDGs) approach, introduced by Khuri [8], can be used for this purpose. It only requires the specification of a parameter space containing the variance components rather than their actual values. The QDGs consist of plots of the maxima and minima, over the parameter space, of the quantiles of the variance component estimators. These plots provide a comprehensive assessment of the quality of estimation of the variance components obtained with a given design. More importantly, they can be utilized to compare several candidate designs for the same model. Comparisons of designs for the one-way and the two-way random models were made by Lee and Khuri [13,14] using the QDGs approach. An application of this approach in a toxicological study can be found in Lee et al. [12].

In this article, we provide one further application of the QDGs approach to comparing three designs for the random three-fold nested model. These designs include the four-stage balanced nested design, the SND, and the IND. The same designs were used in Leone and Nelson [15] and Leone et al. [16]. The designs are compared using ANOVA estimators as well as maximum likelihood (ML) estimators of the variance components. A comparison of ANOVA and ML estimation, using each of the aforementioned designs, is also made.

## 2. ANOVA estimation

Consider the unbalanced three-fold nested random model,

$$y_{ijkl} = \mu + \alpha_i + \beta_{ij} + \gamma_{ijk} + \epsilon_{ijkl}, \tag{1}$$

$i = 1, 2, \ldots, a$; $j = 1, 2, \ldots, b_i$; $k = 1, 2, \ldots, c_{ij}$; $l = 1, 2, \ldots, n_{ijk}$, where $\mu$ is a fixed unknown parameter, $\alpha_i$ is the effect of the $i$-th level of factor A, $\beta_{ij}$ is the effect of the $j$-th level of factor B nested within the $i$-th level of A, $\gamma_{ijk}$ is the effect of the $k$-th level of factor C nested within the $j$-th level of B, which is also nested within the $i$-th level of A, and $\epsilon_{ijkl}$ is a random experimental error. We assume that all the effects in the model are independently distributed, such that $\alpha_i \sim N(0, \sigma_\alpha^2)$, $\beta_{ij} \sim N(0, \sigma_{\beta(\alpha)}^2)$, $\gamma_{ijk} \sim N(0, \sigma_{\gamma(\alpha\beta)}^2)$, and $\epsilon_{ijkl} \sim N(0, \sigma_\epsilon^2)$. Let $b_{\cdot} = \sum_{i=1}^a b_i$, $c_{\cdot\cdot} = \sum_{i=1}^a \sum_{j=1}^{b_i} c_{ij}$, $n_{i\cdot\cdot} = \sum_{j=1}^{b_i} \sum_{k=1}^{c_{ij}} n_{ijk}$, $n_{ij\cdot} = \sum_{k=1}^{c_{ij}} n_{ijk}$, and $N = \sum_{i=1}^a \sum_{j=1}^{b_i} \sum_{k=1}^{c_{ij}} n_{ijk}$. If $b_i = b$ for all $i$, $c_{ij} = c$ for all $i, j$, and $n_{ijk} = n$ for all $i, j, k$, then the model is said to be balanced, and the design is labelled as a balanced nested design.

Model (1) can be written in vector form as

$$\boldsymbol{y} = \mu \mathbf{1}_N + \boldsymbol{X}_1 \boldsymbol{\alpha} + \boldsymbol{X}_2 \boldsymbol{\beta} + \boldsymbol{X}_3 \boldsymbol{\gamma} + \boldsymbol{\epsilon}, \tag{2}$$

where $\mathbf{1}_N$ is the vector of ones of order $N \times 1$, $\boldsymbol{X}_1 = \oplus_{i=1}^a \mathbf{1}_{n_{i\cdot\cdot}}$, $\boldsymbol{X}_2 = \oplus_{i=1}^a \oplus_{j=1}^{b_i} \mathbf{1}_{n_{ij\cdot}}$, $\boldsymbol{X}_3 = \oplus_{i=1}^a \oplus_{j=1}^{b_i} \oplus_{k=1}^{c_{ij}} \mathbf{1}_{n_{ijk}}$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_a)'$, $\boldsymbol{\beta} = (\beta_{11}, \beta_{12}, \ldots, \beta_{1b_1}, \ldots, \beta_{ab_a})'$, and $\boldsymbol{\gamma} = (\gamma_{111}, \gamma_{112}, \ldots, \gamma_{11c_{11}}, \ldots, \gamma_{ab_a c_{ab_a}})'$. Here, the symbol $\oplus$ represents the direct sum of matrices. The error vector $\boldsymbol{\epsilon}$ is defined analogously to $\boldsymbol{y}$, the $N \times 1$ vector of observations. Note that the mean of $\boldsymbol{y}$ is $\mu \mathbf{1}_N$ and its variance–covariance matrix, denoted by $\boldsymbol{\Sigma}$, is of the form

$$\boldsymbol{\Sigma} = \boldsymbol{X}_1 \boldsymbol{X}_1' \sigma_\alpha^2 + \boldsymbol{X}_2 \boldsymbol{X}_2' \sigma_{\beta(\alpha)}^2 + \boldsymbol{X}_3 \boldsymbol{X}_3' \sigma_{\gamma(\alpha\beta)}^2 + \boldsymbol{I}_N \sigma_\epsilon^2.$$

The ANOVA table for model (1) is of the form

| Sources | d.f. | Sum of squares | Expected mean squares |
|---|---|---|---|
| A | $a-1$ | $\mathrm{SS_A}$ | $k_1\sigma_\alpha^2 + k_2\sigma_{\beta(\alpha)}^2 + k_3\sigma_{\gamma(\alpha\beta)}^2 + \sigma_\epsilon^2$ |
| B(A) | $b_{\cdot} - a$ | $\mathrm{SS_{B(A)}}$ | $k_4\sigma_{\beta(\alpha)}^2 + k_5\sigma_{\gamma(\alpha\beta)}^2 + \sigma_\epsilon^2$ |
| C(AB) | $c_{\cdot\cdot} - b_{\cdot}$ | $\mathrm{SS_{C(AB)}}$ | $k_6\sigma_{\gamma(\alpha\beta)}^2 + \sigma_\epsilon^2$ |
| Error | $N - c_{\cdot\cdot}$ | $\mathrm{SS_E}$ | $\sigma_\epsilon^2$ |
| Total (adjusted) | $N-1$ | $\mathrm{SS_T}$ | |

where $\mathrm{SS_A} = \boldsymbol{y}'\boldsymbol{Q}_\mathrm{A}\boldsymbol{y}$, $\mathrm{SS_{B(A)}} = \boldsymbol{y}'\boldsymbol{Q}_\mathrm{B}\boldsymbol{y}$, $\mathrm{SS_{C(AB)}} = \boldsymbol{y}'\boldsymbol{Q}_\mathrm{C}\boldsymbol{y}$, and $\mathrm{SS_E} = \boldsymbol{y}'\boldsymbol{R}\boldsymbol{y}$ with

$$\begin{aligned}
\boldsymbol{Q}_\mathrm{A} &= \oplus_{i=1}^a \left(\frac{1}{n_{i\cdot\cdot}} \boldsymbol{J}_{n_{i\cdot\cdot}}\right) - \frac{1}{N}\boldsymbol{J}_N, \\
\boldsymbol{Q}_\mathrm{B} &= \oplus_{i=1}^a \oplus_{j=1}^{b_i} \left(\frac{1}{n_{ij\cdot}} \boldsymbol{J}_{n_{ij\cdot}}\right) - \oplus_{i=1}^a \left(\frac{1}{n_{i\cdot\cdot}} \boldsymbol{J}_{n_{i\cdot\cdot}}\right), \\
\boldsymbol{Q}_\mathrm{C} &= \oplus_{i=1}^a \oplus_{j=1}^{b_i} \oplus_{k=1}^{c_{ij}} \left(\frac{1}{n_{ijk}} \boldsymbol{J}_{n_{ijk}}\right) - \oplus_{i=1}^a \oplus_{j=1}^{b_i} \left(\frac{1}{n_{ij\cdot}} \boldsymbol{J}_{n_{ij\cdot}}\right), \text{ and} \\
\boldsymbol{R} &= \boldsymbol{I}_\mathrm{N} - \oplus_{i=1}^a \oplus_{j=1}^{b_i} \oplus_{k=1}^{c_{ij}} \left(\frac{1}{n_{ijk}} \boldsymbol{J}_{n_{ijk}}\right), \text{ respectively.}
\end{aligned} \tag{3}$$

The coefficients of the variance components in the expressions for the expected mean squares are [18]

$$k_1 = \frac{1}{a-1} \sum_{i=1}^a n_{i\cdot\cdot}^2 \left(\frac{1}{n_{i\cdot\cdot}} - \frac{1}{N}\right), \quad k_2 = \frac{1}{a-1} \sum_{i=1}^a \sum_{j=1}^{b_i} n_{ij\cdot}^2 \left(\frac{1}{n_{i\cdot\cdot}} - \frac{1}{N}\right),$$

$$k_3 = \frac{1}{a-1}\sum_{i=1}^{a}\sum_{j=1}^{b_i}\sum_{k=1}^{c_{ij}} n_{ijk}^2\left(\frac{1}{n_{i..}} - \frac{1}{N}\right), \quad k_4 = \frac{1}{b. - a}\sum_{i=1}^{a}\sum_{j=1}^{b_i} n_{ij.}^2\left(\frac{1}{n_{ij.}} - \frac{1}{n_{i..}}\right), \tag{4}$$

$$k_5 = \frac{1}{b. - a}\sum_{i=1}^{a}\sum_{j=1}^{b_i}\sum_{k=1}^{c_{ij}} n_{ijk}^2\left(\frac{1}{n_{ij.}} - \frac{1}{n_{i..}}\right), \quad k_6 = \frac{1}{c.. - b.}\sum_{i=1}^{a}\sum_{j=1}^{b_i}\sum_{k=1}^{c_{ij}} n_{ijk}^2\left(\frac{1}{n_{ijk}} - \frac{1}{n_{ij.}}\right).$$

From the expected mean squares given in the ANOVA table, an ANOVA estimator of $\sigma_\epsilon^2$ is $\hat{\sigma}_\epsilon^2 = \boldsymbol{y}'(\boldsymbol{R}/(N - c..))\boldsymbol{y}$, and for $\sigma_\alpha^2$, $\sigma_{\beta(\alpha)}^2$, and $\sigma_{\gamma(\alpha\beta)}^2$, the ANOVA estimators are given by

$$\hat{\sigma}_{\alpha\mathrm{A}}^2 = \boldsymbol{y}'\boldsymbol{Q}_\alpha\boldsymbol{y}, \quad \hat{\sigma}_{\beta(\alpha)\mathrm{A}}^2 = \boldsymbol{y}'\boldsymbol{Q}_\beta\boldsymbol{y}, \quad \text{and } \hat{\sigma}_{\gamma(\alpha\beta)\mathrm{A}}^2 = \boldsymbol{y}'\boldsymbol{Q}_\gamma\boldsymbol{y},$$

respectively, where

$$\begin{aligned}
\boldsymbol{Q}_\alpha &= \frac{1}{k_1(a-1)}\boldsymbol{Q}_\mathrm{A} - \frac{k_2}{k_1k_4(b. - a)}\boldsymbol{Q}_\mathrm{B} - \frac{(k_3k_4 - k_2k_5)}{k_1k_4k_6(c.. - b.)}\boldsymbol{Q}_\mathrm{C} - \frac{(k_2k_5 + k_4k_6 - k_3k_4 - k_2k_6)}{k_1k_4k_6(N - c..)}\boldsymbol{R},\\
\boldsymbol{Q}_\beta &= \frac{1}{k_4(b. - a)}\boldsymbol{Q}_\mathrm{B} - \frac{k_5}{k_4k_6(c.. - b.)}\boldsymbol{Q}_\mathrm{C} - \frac{(k_6 - k_5)}{k_4k_6(N - c..)}\boldsymbol{R}, \quad \text{and}\\
\boldsymbol{Q}_\gamma &= \frac{1}{k_6(c.. - b.)}\boldsymbol{Q}_\mathrm{C} - \frac{1}{k_6(N - c..)}\boldsymbol{R}.
\end{aligned} \tag{5}$$

Since $\hat{\sigma}_{\alpha\mathrm{A}}^2$, $\hat{\sigma}_{\beta(\alpha)\mathrm{A}}^2$, and $\hat{\sigma}_{\gamma(\alpha\beta)\mathrm{A}}^2$ are scale-dependent, we consider instead their scaled versions, namely, $\hat{W}_{\alpha\mathrm{A}} = (\hat{\sigma}_{\alpha\mathrm{A}}^2/\sigma_\alpha^2)$, $\hat{W}_{\beta(\alpha)\mathrm{A}} = (\hat{\sigma}_{\beta(\alpha)\mathrm{A}}^2/\sigma_{\beta(\alpha)}^2)$, and $\hat{W}_{\gamma(\alpha\beta)\mathrm{A}} = (\hat{\sigma}_{\gamma(\alpha\beta)\mathrm{A}}^2/\sigma_{\gamma(\alpha\beta)}^2)$, respectively. A theorem in Johnson and Kotz ([5], p. 153) allows us to express these scaled estimators as linear combinations of independent central chi-squared variates of the form

$$\hat{W}_{\alpha\mathrm{A}} \overset{\mathrm{D}}{=} \sum_{j=1}^{f}\gamma_j\,\chi_{l_j}^2, \quad \hat{W}_{\beta(\alpha)\mathrm{A}} \overset{\mathrm{D}}{=} \sum_{j=1}^{g}\delta_j\,\chi_{m_j}^2, \quad \text{and } \hat{W}_{\gamma(\alpha\beta)\mathrm{A}} \overset{\mathrm{D}}{=} \sum_{j=1}^{h}\xi_j\,\chi_{\nu_j}^2, \tag{6}$$

where $f$, $g$, and $h$ are the numbers of distinct positive eigenvalues of $\boldsymbol{Q}_\alpha\boldsymbol{\Sigma}_\alpha$, $\boldsymbol{Q}_\beta\boldsymbol{\Sigma}_\beta$, and $\boldsymbol{Q}_\gamma\boldsymbol{\Sigma}_\gamma$, respectively, and $\gamma_j$, $\delta_j$, and $\xi_j$ are their $j$-th positive eigenvalues with multiplicities $l_j$, $m_j$, and $\nu_j$, respectively. The matrices $\boldsymbol{\Sigma}_\alpha$, $\boldsymbol{\Sigma}_\beta$, and $\boldsymbol{\Sigma}_\gamma$ are given by

$$\begin{aligned}
\boldsymbol{\Sigma}_\alpha &= \frac{1}{\sigma_\alpha^2}\boldsymbol{\Sigma} = \boldsymbol{X}_1\boldsymbol{X}_1' + \boldsymbol{X}_2\boldsymbol{X}_2'\frac{\eta_2}{\eta_1} + \boldsymbol{X}_3\boldsymbol{X}_3'\frac{\eta_3}{\eta_1} + \boldsymbol{I}_N\frac{1}{\eta_1},\\
\boldsymbol{\Sigma}_\beta &= \frac{1}{\sigma_{\beta(\alpha)}^2}\boldsymbol{\Sigma} = \boldsymbol{X}_1\boldsymbol{X}_1'\frac{\eta_1}{\eta_2} + \boldsymbol{X}_2\boldsymbol{X}_2' + \boldsymbol{X}_3\boldsymbol{X}_3'\frac{\eta_3}{\eta_2} + \boldsymbol{I}_N\frac{1}{\eta_2}, \quad \text{and}\\
\boldsymbol{\Sigma}_\gamma &= \frac{1}{\sigma_{\gamma(\alpha\beta)}^2}\boldsymbol{\Sigma} = \boldsymbol{X}_1\boldsymbol{X}_1'\frac{\eta_1}{\eta_3} + \boldsymbol{X}_2\boldsymbol{X}_2'\frac{\eta_2}{\eta_3} + \boldsymbol{X}_3\boldsymbol{X}_3' + \boldsymbol{I}_N\frac{1}{\eta_3},
\end{aligned} \tag{7}$$

where $\eta_1 = (\sigma_\alpha^2/\sigma_\epsilon^2)$, $\eta_2 = (\sigma_{\beta(\alpha)}^2/\sigma_\epsilon^2)$, and $\eta_3 = (\sigma_{\gamma(\alpha\beta)}^2/\sigma_\epsilon^2)$.

The exact distributions of $\hat{W}_{\alpha\mathrm{A}}$, $\hat{W}_{\beta(\alpha)\mathrm{A}}$, and $\hat{W}_{\gamma(\alpha\beta)\mathrm{A}}$ are determined by their quantiles, which can be obtained by using, for example, Davies' [4] algorithm. In the special case of the balanced nested design, these distributions are of the form

$$\begin{aligned}
\hat{W}_{\alpha\mathrm{A}} &\overset{\mathrm{D}}{=} \frac{1}{(a-1)bcn}\left[bcn + cn\frac{\eta_2}{\eta_1} + n\frac{\eta_3}{\eta_1} + \frac{1}{\eta_1}\right]\chi^2_{a-1}, \\
&\quad - \frac{1}{ab(b-1)cn}\left[cn\frac{\eta_2}{\eta_1} + n\frac{\eta_3}{\eta_1} + \frac{1}{\eta_1}\right]\chi^2_{a(b-1)}, \\
\hat{W}_{\beta(\alpha)\mathrm{A}} &\overset{\mathrm{D}}{=} \frac{1}{a(b-1)cn}\left[cn + n\frac{\eta_3}{\eta_2} + \frac{1}{\eta_2}\right]\chi^2_{a(b-1)} \\
&\quad - \frac{1}{abc(c-1)n}\left[n\frac{\eta_3}{\eta_2} + \frac{1}{\eta_2}\right]\chi^2_{ab(c-1)}, \text{ and} \\
\hat{W}_{\gamma(\alpha\beta)\mathrm{A}} &\overset{\mathrm{D}}{=} \frac{1}{ab(c-1)n}\left(n + \frac{1}{\eta_3}\right)\chi^2_{ab(c-1)} - \frac{1}{abcn(n-1)}\left(\frac{1}{\eta_3}\right)\chi^2_{abc(n-1)}.
\end{aligned} \tag{8}$$

This follows from the fact that, for model (1), the expected mean squares of the factors are $\mathrm{E(MS_A)} = bcn\sigma^2_\alpha + cn\sigma^2_{\beta(\alpha)} + n\sigma^2_{\gamma(\alpha\beta)} + \sigma^2_\epsilon$, $\mathrm{E(MS_{B(A)})} = cn\sigma^2_{\beta(\alpha)} + n\sigma^2_{\gamma(\alpha\beta)} + \sigma^2_\epsilon$, $\mathrm{E(MS_{C(AB)})} = n\sigma^2_{\gamma(\alpha\beta)} + \sigma^2_\epsilon$, and $\mathrm{E(MS_E)} = \sigma^2_\epsilon$. Furthermore, all four sums of squares are independently distributed such that $\mathrm{SS_A/E(MS_A)} \sim \chi^2_{a-1}$, $\mathrm{SS_{B(A)}/E(MS_{B(A)})} \sim \chi^2_{a(b-1)}$, $\mathrm{SS_{C(AB)}/E(MS_{C(AB)})} \sim \chi^2_{ab(c-1)}$, and $\mathrm{SS_E}/\sigma^2_\epsilon \sim \chi^2_{abc(n-1)}$.

## 3. Comparisons of designs with ANOVA estimation

The quantiles of $\hat{W}_{\alpha\mathrm{A}}$, $\hat{W}_{\beta(\alpha)\mathrm{A}}$, and $\hat{W}_{\gamma(\alpha\beta)\mathrm{A}}$ depend on the unknown values of $\eta_1 = (\sigma^2_\alpha/\sigma^2_\epsilon)$, $\eta_2 = (\sigma^2_{\beta(\alpha)}/\sigma^2_\epsilon)$, and $\eta_3 = (\sigma^2_{\gamma(\alpha\beta)}/\sigma^2_\epsilon)$, as well as on the chosen design, which we denote by D. Let $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$. The dependence of the $p$-th quantile on $\boldsymbol{\eta}$ and D is indicated by writing $q^\alpha_\mathrm{D}(p, \boldsymbol{\eta})$ for $\hat{W}_{\alpha\mathrm{A}}$, $q^\beta_\mathrm{D}(p, \boldsymbol{\eta})$ for $\hat{W}_{\beta(\alpha)\mathrm{A}}$, and $q^\gamma_\mathrm{D}(p, \boldsymbol{\eta})$ for $\hat{W}_{\gamma(\alpha\beta)\mathrm{A}}$. Let S denote a specified region in the parameter space of $\boldsymbol{\eta}$. The QDGs of $\hat{W}_{\alpha\mathrm{A}}$, $\hat{W}_{\beta(\alpha)\mathrm{A}}$, and $\hat{W}_{\gamma(\alpha\beta)\mathrm{A}}$ are then obtained by plotting their minimum and maximum quantiles over S against $p$ [8]. We thus have
for $\hat{W}_{\alpha\mathrm{A}}$,

$$Q^{\min}_{\mathrm{D},\alpha}(p) = \min_{\boldsymbol{\eta}\in\mathrm{S}} q^\alpha_\mathrm{D}(p, \boldsymbol{\eta}) \quad \text{and} \quad Q^{\max}_{\mathrm{D},\alpha}(p) = \max_{\boldsymbol{\eta}\in S} q^\alpha_\mathrm{D}(p, \boldsymbol{\eta}), \tag{9}$$

for $\hat{W}_{\beta(\alpha)\mathrm{A}}$,

$$Q^{\min}_{\mathrm{D},\beta}(p) = \min_{\boldsymbol{\eta}\in\mathrm{S}} q^\beta_\mathrm{D}(p, \boldsymbol{\eta}) \quad \text{and} \quad Q^{\max}_{\mathrm{D},\beta}(p) = \max_{\boldsymbol{\eta}\in\mathrm{S}} q^\beta_\mathrm{D}(p, \boldsymbol{\eta}), \tag{10}$$

and for $\hat{W}_{\gamma(\alpha\beta)\mathrm{A}}$,

$$Q^{\min}_{\mathrm{D},\gamma}(p) = \min_{\boldsymbol{\eta}\in\mathrm{S}} q^\gamma_\mathrm{D}(p, \boldsymbol{\eta}) \quad \text{and} \quad Q^{\max}_{\mathrm{D},\gamma}(p) = \max_{\boldsymbol{\eta}\in\mathrm{S}} q^\gamma_\mathrm{D}(p, \boldsymbol{\eta}). \tag{11}$$

Some guidelines for comparing designs using QDGs can be summarized as follows: First, a design $\mathrm{D}_1$ is considered to be more 'robust' in estimating variance components than another design $\mathrm{D}_2$ if the values of $Q^{\max}_\mathrm{D}(p) - Q^{\min}_\mathrm{D}(p)$ are smaller for $\mathrm{D}_1$ than they are for $\mathrm{D}_2$ over the range of $p$ ($0 \le p \le 1$). Also, $\mathrm{D}_1$ is considered more 'efficient' than $\mathrm{D}_2$ if the absolute values of $Q^{\min}_\mathrm{D}(p)$ and $Q^{\max}_\mathrm{D}(p)$ over the range of $p$ are smaller for $\mathrm{D}_1$ than they are for $\mathrm{D}_2$ (i.e. the QDGs for $\mathrm{D}_1$ are flatter than those for $\mathrm{D}_2$) [8, 12].

In this study, three designs, namely the four-stage balanced, staggered, as well as inverted nested designs, are compared. The total number of observations for each design is set to 40. The same designs were studied by Leone and his co-workers [15,16]. The parameter space for $\boldsymbol{\eta}$ considered here is also the same as theirs. We denote these three designs by $\mathrm{D_{bal}}$, $\mathrm{D_{stg}}$, and $\mathrm{D_{inv}}$, respectively.

### 3.1. *Distributions of ANOVA estimators for nested designs*

For a sample of size 40, the expected mean squares and corresponding degrees of freedom for the balanced, staggered, and inverted nested designs considered in Leone et al. [16] are given in their Table 1 (see p. 721).

For the balanced nested design, $D_{\text{bal}}$, where $a = 5$, $b = 2$, $c = 2$, and $n = 2$, the scaled ANOVA estimators for $\sigma_\alpha^2$, $\sigma_{\beta(\alpha)}^2$, and $\sigma_{\gamma(\alpha\beta)}^2$ are distributed as (see Equation (8)),

$$\begin{aligned}
\hat{W}_{\alpha\mathrm{A}} &\overset{\mathrm{D}}{=} \frac{1}{32}\left[8+4\left(\frac{\eta_2}{\eta_1}\right)+2\left(\frac{\eta_3}{\eta_1}\right)+\frac{1}{\eta_1}\right]\chi_4^2 - \frac{1}{40}\left[4\left(\frac{\eta_2}{\eta_1}\right)+2\left(\frac{\eta_3}{\eta_1}\right)+\frac{1}{\eta_1}\right]\chi_5^2,\\
\hat{W}_{\beta(\alpha)\mathrm{A}} &\overset{\mathrm{D}}{=} \frac{1}{20}\left[4+2\left(\frac{\eta_3}{\eta_2}\right)+\frac{1}{\eta_2}\right]\chi_5^2 - \frac{1}{40}\left[2\left(\frac{\eta_3}{\eta_2}\right)+\frac{1}{\eta_2}\right]\chi_{10}^2, \text{ and}\\
\hat{W}_{\gamma(\alpha\beta)\mathrm{A}} &\overset{\mathrm{D}}{=} \frac{1}{20}\left[2+\frac{1}{\eta_3}\right]\chi_{10}^2 - \frac{1}{40\eta_3}\chi_{20}^2.
\end{aligned} \tag{12}$$

For the staggered nested design, $D_{\text{stg}}$, we have $a = 10$, and for each $i$ $(i = 1, 2, \ldots, a)$, $b_i = 2$, and $c_{ij}$ and $n_{ijk}$ are given by

$$c_{ij} = \begin{cases} 2 & \text{if } j = 1 \\ 1 & \text{if } j = 2, \end{cases}$$

$$n_{ijk} = \begin{cases} 2 & \text{if } j = 1, k = 1 \\ 1 & \text{if } j = 1, k = 2 \text{ or if } j = 2, k = 1. \end{cases}$$

Hence, $b_. = 20$, $c_{..} = 30$, and $N = 40$. In this case, model (2) can be written as

$$\boldsymbol{y} = \mu\mathbf{1}_{40} + (\boldsymbol{I}_{10} \otimes \mathbf{1}_4)\boldsymbol{\alpha} + (\boldsymbol{I}_{10} \otimes \boldsymbol{A})\boldsymbol{\beta} + (\boldsymbol{I}_{10} \otimes \boldsymbol{B})\boldsymbol{\gamma} + \boldsymbol{\epsilon},$$

where $\boldsymbol{A} = \mathbf{1}_3 \oplus \boldsymbol{I}_1$ and $\boldsymbol{B} = \mathbf{1}_2 \oplus \boldsymbol{I}_2$. In the above model, $\otimes$ denotes the Kronecker (direct) product of matrices. The variance–covariance matrix of $\boldsymbol{y}$, $\boldsymbol{\Sigma}$, is

$$\boldsymbol{\Sigma} = (\boldsymbol{I}_{10} \otimes \boldsymbol{J}_4)\sigma_\alpha^2 + (\boldsymbol{I}_{10} \otimes \boldsymbol{AA}')\sigma_{\beta(\alpha)}^2 + (\boldsymbol{I}_{10} \otimes \boldsymbol{BB}')\sigma_{\gamma(\alpha\beta)}^2 + \boldsymbol{I}_{40}\sigma_\epsilon^2,$$

and hence, $\boldsymbol{\Sigma}_\alpha$, $\boldsymbol{\Sigma}_\beta$, and $\boldsymbol{\Sigma}_\gamma$ in Equation (7) are given by

$$\begin{aligned}
\boldsymbol{\Sigma}_\alpha &= (\boldsymbol{I}_{10} \otimes \boldsymbol{J}_4) + (\boldsymbol{I}_{10} \otimes \boldsymbol{AA}')\frac{\eta_2}{\eta_1} + (\boldsymbol{I}_{10} \otimes \boldsymbol{BB}')\frac{\eta_3}{\eta_1} + \boldsymbol{I}_{40}\frac{1}{\eta_1},\\
\boldsymbol{\Sigma}_\beta &= (\boldsymbol{I}_{10} \otimes \boldsymbol{J}_4)\frac{\eta_1}{\eta_2} + (\boldsymbol{I}_{10} \otimes \boldsymbol{AA}') + (\boldsymbol{I}_{10} \otimes \boldsymbol{BB}')\frac{\eta_3}{\eta_2} + \boldsymbol{I}_{40}\frac{1}{\eta_2}, \text{ and}\\
\boldsymbol{\Sigma}_\gamma &= (\boldsymbol{I}_{10} \otimes \boldsymbol{J}_4)\frac{\eta_1}{\eta_3} + (\boldsymbol{I}_{10} \otimes \boldsymbol{AA}')\frac{\eta_2}{\eta_3} + (\boldsymbol{I}_{10} \otimes \boldsymbol{BB}') + \boldsymbol{I}_{40}\frac{1}{\eta_3},
\end{aligned}$$

respectively. On the basis of the expected mean squares given in Table 1 in Leone et al. [16], the matrices $\boldsymbol{Q}_\alpha$, $\boldsymbol{Q}_\beta$, and $\boldsymbol{Q}_\gamma$ in Equation (5) for $D_{\text{stg}}$ are $\boldsymbol{Q}_\alpha = (1/36)\boldsymbol{Q}_{\mathrm{A}} - (1/24)\boldsymbol{Q}_B + (1/120)\boldsymbol{Q}_{\mathrm{C}} + (1/120)\boldsymbol{R}$, $\boldsymbol{Q}_\beta = (1/15)\boldsymbol{Q}_B - (7/120)\boldsymbol{Q}_{\mathrm{C}} - (1/120)\boldsymbol{R}$, and $\boldsymbol{Q}_\gamma = (3/40)\boldsymbol{Q}_{\mathrm{C}} - (3/40)\boldsymbol{R}$, where $\boldsymbol{Q}_{\mathrm{A}} = (1/4)(\boldsymbol{I}_{10} \otimes \boldsymbol{J}_4) - (1/40)\boldsymbol{J}_{40}$, $\boldsymbol{Q}_B = \boldsymbol{I}_{10} \otimes \boldsymbol{A}^* - (1/4)(\boldsymbol{I}_{10} \otimes \boldsymbol{J}_4)$, $\boldsymbol{Q}_{\mathrm{C}} = \boldsymbol{I}_{10} \otimes \boldsymbol{B}^* - \boldsymbol{I}_{10} \otimes \boldsymbol{A}^*$, and $\boldsymbol{R} = \boldsymbol{I}_{40} - \boldsymbol{I}_{10} \otimes \boldsymbol{B}^*$ with $\boldsymbol{A}^* = ((1/3)\boldsymbol{J}_3) \oplus \boldsymbol{I}_1$ and $\boldsymbol{B}^* = ((1/2)\boldsymbol{J}_2) \oplus \boldsymbol{I}_2$. Note that the matrices $\boldsymbol{Q}_{\mathrm{A}}, \boldsymbol{Q}_{\mathrm{B}}, \boldsymbol{Q}_{\mathrm{C}}$, and $\boldsymbol{R}$ were obtained from Equation (3). Using the matrices, $\boldsymbol{Q}_\alpha$, $\boldsymbol{Q}_\beta$, $\boldsymbol{Q}_\gamma$, $\boldsymbol{\Sigma}_\alpha$, $\boldsymbol{\Sigma}_\beta$, and $\boldsymbol{\Sigma}_\gamma$, the distributions of $\hat{W}_{\alpha\mathrm{A}}$, $\hat{W}_{\beta(\alpha)\mathrm{A}}$, and $\hat{W}_{\gamma(\alpha\beta)\mathrm{A}}$ for design $D_{\text{stg}}$ can be obtained from Equation (6).

Table 1. Minimum and maximum quantile values of $\hat{W}_{\alpha\mathrm{A}}$, $\hat{W}_{\beta(\alpha)\mathrm{A}}$, and $\hat{W}_{\gamma(\alpha\beta)\mathrm{A}}$ for selected values of $p$ for designs $D_{\mathrm{bal}}$, $D_{\mathrm{stg}}$, and $D_{\mathrm{inv}}$.

| | $\hat{W}_{\alpha A}$ | | | | | | $\hat{W}_{\beta(\alpha)A}$ | | | | | | $\hat{W}_{\gamma(\alpha\beta)\mathrm{A}}$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $D_{\mathrm{bal}}$ | | $D_{\mathrm{stg}}$ | | $D_{\mathrm{inv}}$ | | $D_{\mathrm{bal}}$ | | $D_{\mathrm{stg}}$ | | $D_{\mathrm{inv}}$ | | $D_{\mathrm{bal}}$ | | $D_{\mathrm{stg}}$ | | $D_{\mathrm{inv}}$ | |
| $p$ | Min. | Max. | Min. | Max. | Min. | Max. | Min. | Max. | Min. | Max. | Min. | Max. | Min. | Max. | Min. | Max. | Min. | Max. |
| 0.01 | −15.503 | −0.037 | −14.477 | 0.098 | −13.543 | 0.191 | −7.572 | 0.030 | −11.035 | 0.115 | −13.171 | 0.133 | −0.210 | 0.213 | −0.663 | 0.187 | −1.528 | 0.104 |
| 0.05 | −9.836 | 0.092 | −9.379 | 0.271 | −8.750 | 0.360 | −5.054 | 0.163 | −7.143 | 0.288 | −8.275 | 0.315 | 0.048 | 0.360 | −0.226 | 0.340 | −0.689 | 0.267 |
| 0.10 | −7.201 | 0.192 | −6.898 | 0.383 | −6.405 | 0.464 | −3.806 | 0.265 | −5.240 | 0.399 | −5.943 | 0.427 | 0.204 | 0.458 | 0.004 | 0.442 | −0.299 | 0.374 |
| 0.20 | −4.323 | 0.355 | −4.081 | 0.540 | −3.730 | 0.607 | −2.346 | 0.425 | −3.064 | 0.555 | −3.345 | 0.581 | 0.416 | 0.597 | 0.292 | 0.585 | 0.130 | 0.529 |
| 0.30 | −2.424 | 0.506 | −2.149 | 0.671 | −1.891 | 0.723 | −1.297 | 0.568 | −1.557 | 0.683 | −1.601 | 0.705 | 0.587 | 0.712 | 0.513 | 0.704 | 0.425 | 0.658 |
| 0.40 | −0.875 | 0.660 | −0.541 | 0.794 | −0.363 | 0.830 | −0.372 | 0.710 | −0.291 | 0.804 | −0.175 | 0.820 | 0.747 | 0.820 | 0.712 | 0.816 | 0.679 | 0.782 |
| 0.50 | 0.572 | 0.826 | 0.881 | 0.974 | 0.930 | 1.044 | 0.542 | 0.861 | 0.893 | 0.926 | 0.933 | 1.122 | 0.908 | 0.931 | 0.910 | 0.929 | 0.906 | 0.926 |
| 0.60 | 1.015 | 2.096 | 1.056 | 2.443 | 1.055 | 2.444 | 1.030 | 1.524 | 1.058 | 2.093 | 1.061 | 2.404 | 1.050 | 1.081 | 1.052 | 1.121 | 1.049 | 1.186 |
| 0.70 | 1.244 | 3.865 | 1.214 | 4.075 | 1.189 | 3.951 | 1.232 | 2.666 | 1.211 | 3.412 | 1.204 | 3.780 | 1.188 | 1.280 | 1.194 | 1.361 | 1.211 | 1.480 |
| 0.80 | 1.548 | 6.159 | 1.415 | 6.054 | 1.358 | 5.747 | 1.498 | 4.135 | 1.404 | 5.019 | 1.384 | 5.420 | 1.364 | 1.533 | 1.374 | 1.662 | 1.420 | 1.851 |
| 0.90 | 2.040 | 9.787 | 1.725 | 8.970 | 1.618 | 8.342 | 1.920 | 6.435 | 1.700 | 7.394 | 1.658 | 7.792 | 1.632 | 1.918 | 1.650 | 2.119 | 1.744 | 2.414 |
| 0.95 | 2.509 | 13.210 | 2.008 | 11.560 | 1.855 | 10.605 | 2.317 | 8.580 | 1.970 | 9.503 | 1.907 | 9.862 | 1.877 | 2.268 | 1.902 | 2.533 | 2.042 | 2.928 |
| 0.99 | 3.548 | 20.744 | 2.609 | 16.928 | 2.359 | 15.230 | 3.188 | 13.247 | 2.540 | 13.869 | 2.430 | 14.082 | 2.395 | 3.007 | 2.433 | 3.401 | 2.684 | 4.017 |

For the inverted design, $D_{inv}$, we have $a = 16$, and if the values of $b_i (i = 1, 2, \ldots, 16)$ are represented by the vector $\boldsymbol{b}$, then $\boldsymbol{b} = \mathbf{1}_4 \otimes \boldsymbol{b}_1$, where $\boldsymbol{b}_1 = (2, 2, 2, 1)'$. Furthermore, if the values of $c_{ij}$ and $n_{ijk}$ are represented by the vectors $\boldsymbol{c}$ and $\boldsymbol{n}$, respectively, then $\boldsymbol{c} = \mathbf{1}_4 \otimes \boldsymbol{c}_1$ and $\boldsymbol{n} = \mathbf{1}_4 \otimes \boldsymbol{n}_1$, where $\boldsymbol{c}_1 = (2, 1, 2, 1, 1, 1, 1)'$ and $\boldsymbol{n}_1 = (2, 1, 1, 1, 1, 1, 1, 1, 1)'$. Hence, $b_. = 28$, $c_{..} = 36$, and $N = 40$. Model (2) can then be expressed as

$$\boldsymbol{y} = \mu\mathbf{1}_{40} + (\boldsymbol{I}_4 \otimes \boldsymbol{C})\boldsymbol{\alpha} + (\boldsymbol{I}_4 \otimes \boldsymbol{D})\boldsymbol{\beta} + (\boldsymbol{I}_4 \otimes \boldsymbol{E})\boldsymbol{\gamma} + \boldsymbol{\epsilon},$$

where $\boldsymbol{C} = \mathbf{1}_4 \oplus \mathbf{1}_3 \oplus \mathbf{1}_2 \oplus \mathbf{1}_1$, $\boldsymbol{D} = \mathbf{1}_3 \oplus \mathbf{1}_1 \oplus \mathbf{1}_2 \oplus \boldsymbol{I}_4$, and $\boldsymbol{E} = \mathbf{1}_2 \oplus \boldsymbol{I}_8$. The variance–covariance matrix of $\boldsymbol{y}$, $\boldsymbol{\Sigma}$, is

$$\boldsymbol{\Sigma} = (\boldsymbol{I}_4 \otimes \boldsymbol{CC}')\sigma_\alpha^2 + (\boldsymbol{I}_4 \otimes \boldsymbol{DD}')\sigma_{\beta(\alpha)}^2 + (\boldsymbol{I}_4 \otimes \boldsymbol{EE}')\sigma_{\gamma(\alpha\beta)}^2 + \boldsymbol{I}_{40}\sigma_\epsilon^2,$$

and from Equation (7),

$$\boldsymbol{\Sigma}_\alpha = (\boldsymbol{I}_4 \otimes \boldsymbol{CC}') + (\boldsymbol{I}_4 \otimes \boldsymbol{DD}')\frac{\eta_2}{\eta_1} + (\boldsymbol{I}_4 \otimes \boldsymbol{EE}')\frac{\eta_3}{\eta_1} + \boldsymbol{I}_{40}\frac{1}{\eta_1},$$

$$\boldsymbol{\Sigma}_\beta = (\boldsymbol{I}_4 \otimes \boldsymbol{CC}')\frac{\eta_1}{\eta_2} + (\boldsymbol{I}_4 \otimes \boldsymbol{DD}') + (\boldsymbol{I}_4 \otimes \boldsymbol{EE}')\frac{\eta_3}{\eta_2} + \boldsymbol{I}_{40}\frac{1}{\eta_2}, \text{ and}$$

$$\boldsymbol{\Sigma}_\gamma = (\boldsymbol{I}_4 \otimes \boldsymbol{CC}')\frac{\eta_1}{\eta_3} + (\boldsymbol{I}_4 \otimes \boldsymbol{DD}')\frac{\eta_2}{\eta_3} + (\boldsymbol{I}_4 \otimes \boldsymbol{EE}') + \boldsymbol{I}_{40}\frac{1}{\eta_3}.$$

Also, the matrices $\boldsymbol{Q}_\alpha$, $\boldsymbol{Q}_\beta$, and $\boldsymbol{Q}_\gamma$ in Equation (5) for $D_{inv}$ are $\boldsymbol{Q}_\alpha = (1/37)\boldsymbol{Q}_A - (343/8510)\boldsymbol{Q}_B + (103/17020)\boldsymbol{Q}_C + (127/17020)\boldsymbol{R}, \boldsymbol{Q}_\beta = (3/46)\boldsymbol{Q}_B - (57/644)\boldsymbol{Q}_C - (3/161)\boldsymbol{R}$, and $\boldsymbol{Q}_\gamma = (3/28)\boldsymbol{Q}_C - (3/14)\boldsymbol{R}$, where, from Equation (3), $\boldsymbol{Q}_A = \boldsymbol{I}_4 \otimes \boldsymbol{C}^* - (1/40)\boldsymbol{J}_{40}$, $\boldsymbol{Q}_B = \boldsymbol{I}_4 \otimes \boldsymbol{D}^* - \boldsymbol{I}_4 \otimes \boldsymbol{C}^*$, $\boldsymbol{Q}_C = \boldsymbol{I}_4 \otimes \boldsymbol{E}^* - \boldsymbol{I}_4 \otimes \boldsymbol{D}^*$, and $\boldsymbol{R} = \boldsymbol{I}_{40} - \boldsymbol{I}_4 \otimes \boldsymbol{E}^*$. Here, $\boldsymbol{C}^* = (1/4)\boldsymbol{J}_4 \oplus (1/3)\boldsymbol{J}_3 \oplus (1/2)\boldsymbol{J}_2 \oplus \boldsymbol{I}_1$, $\boldsymbol{D}^* = (1/3)\boldsymbol{J}_3 \oplus \boldsymbol{I}_1 \oplus (1/2)\boldsymbol{J}_2 \oplus \boldsymbol{I}_4$, and $\boldsymbol{E}^* = (1/2)\boldsymbol{J}_2 \oplus \boldsymbol{I}_8$. The distributions of $\hat{W}_{\alpha A}$, $\hat{W}_{\beta(\alpha)A}$, and $\hat{W}_{\gamma(\alpha\beta)A}$ for design $D_{inv}$ are then obtained from Equation (6) using the matrices $\boldsymbol{Q}_\alpha$, $\boldsymbol{Q}_\beta$, $\boldsymbol{Q}_\gamma$, $\boldsymbol{\Sigma}_\alpha$, $\boldsymbol{\Sigma}_\beta$, and $\boldsymbol{\Sigma}_\gamma$.

### 3.2. *Comparison of designs*

Davies' [4] algorithm is used to compute the quantiles of $\hat{W}_{\alpha A}$, $\hat{W}_{\beta(\alpha)A}$, and $\hat{W}_{\gamma(\alpha\beta)A}$ for designs $D_{bal}$, $D_{stg}$, and $D_{inv}$. For each design, values of the minimum and the maximum quantiles of the scaled ANOVA estimators, as shown in Equations (9) through (11), are obtained over a region of the parameter space, namely S, for selected values of $p$. The parameter space of interest used in this study is

$$S = \{(\eta_1, \eta_2, \eta_3) \mid 1 \le \eta_1 \le 9,\ 1 \le \eta_2 \le 9,\ 1 \le \eta_3 \le 9\},$$

which covers, as we mentioned earlier, the parameter space considered in Leone et al. [16]. More specifically, for a given $p$ and a given design D, quantile values, $q_D(p, \boldsymbol{\eta})$, are computed using the following values of $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$; $\eta_1 = 1(1)9, \eta_2 = 1(1)9, \eta_3 = 1(1)9$. The resulting minimum and maximum values are shown in Table 1. The QDGs for the designs are constructed on the basis of Table 1 (Figure 1) from which the following observations can be made;

(1) The variabilities in the quantile values of $\hat{W}_{\alpha A}$ with respect to $\boldsymbol{\eta}$ for the three designs are quite similar (Figure 1a). The variability for designs $D_{stg}$ or $D_{inv}$ is slightly smaller than that for design $D_{bal}$, but the difference is not that apparent. Thus, the three designs are comparable to one another in terms of consistency in the ANOVA estimate of $\sigma_\alpha^2$. Moreover, the designs $D_{stg}$ and $D_{inv}$ provide more stable estimates for $\sigma_\alpha^2$ than the design $D_{bal}$.

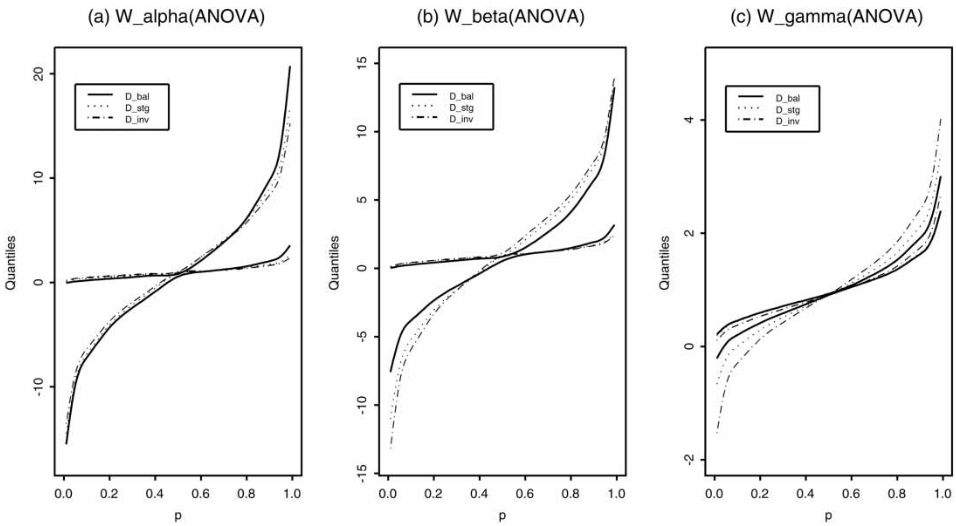


Figure 1. Quantile dispersion graphs of (a) $\hat{W}_{\alpha A} = \hat{\sigma}^2_{\alpha A}/\sigma^2_{\alpha}$, (b) $\hat{W}_{\beta(\alpha)A} = \hat{\sigma}^2_{\beta(\alpha)A}/\sigma^2_{\beta(\alpha)}$, and (c) $\hat{W}_{\gamma(\alpha\beta)A} = \hat{\sigma}^2_{\gamma(\alpha\beta)A}/\sigma^2_{\gamma(\alpha\beta)}$ for designs, $D_{bal}$, $D_{stg}$, and $D_{inv}$, where $\hat{\sigma}^2_{\alpha A}$, $\hat{\sigma}^2_{\beta(\alpha)A}$, and $\hat{\sigma}^2_{\gamma(\alpha\beta)A}$ are the analysis of variance estimates of the variance components.

(2) The variability in the quantile values of $\hat{W}_{\beta(\alpha)A}$ with respect to $\boldsymbol{\eta}$ for design $D_{bal}$ is smaller than those for designs $D_{stg}$ and $D_{inv}$ (Figure 1b). Design $D_{stg}$ shows similar variability to design $D_{inv}$. This indicates that a more precise estimate of $\sigma^2_{\beta(\alpha)}$ is obtained with $D_{bal}$ than with either $D_{stg}$ or $D_{inv}$.

(3) The variability in quantile values of $\hat{W}_{\gamma(\alpha\beta)A}$ with respect to $\boldsymbol{\eta}$ is also smaller with design $D_{bal}$ than with designs $D_{stg}$ and $D_{inv}$ (Figure 1c). The difference between $D_{bal}$ and $D_{inv}$ is noticeable, and $D_{stg}$ provides smaller variability than $D_{inv}$. The degree of uniformity in the quantile values of $\hat{W}_{\gamma(\alpha\beta)A}$ is also larger for $D_{bal}$ than for $D_{inv}$. This indicates that a more stable estimate of $\sigma^2_{\gamma(\alpha\beta)}$ is obtained with $D_{bal}$ than with either $D_{stg}$ or $D_{inv}$, and a more stable estimate is obtained with $D_{stg}$ than with $D_{inv}$.

In summary, the comparability of designs $D_{stg}$ and $D_{inv}$ to design $D_{bal}$ is apparent only when the ANOVA estimate of $\sigma^2_{\alpha}$ is of interest. This is expected since both $D_{stg}$ and $D_{inv}$ have larger degrees of freedom for factor A than $D_{bal}$. The benefit of using $D_{stg}$ or $D_{inv}$ over $D_{bal}$, however, is not large enough to justify losing balance. Thus, if getting a stable ANOVA estimate of $\sigma^2_{\alpha}$ is of major concern, then $D_{stg}$ or $D_{inv}$ are recommended, otherwise, the balanced design is preferred for estimating $\sigma^2_{\beta(\alpha)}$ or $\sigma^2_{\gamma(\alpha\beta)}$.

## 4. Comparisons of designs with ML estimation

Let us denote the ML estimates of $\sigma^2_{\alpha}$, $\sigma^2_{\beta(\alpha)}$, and $\sigma^2_{\gamma(\alpha\beta)}$ by $\hat{\sigma}^2_{\alpha M}$, $\hat{\sigma}^2_{\beta(\alpha)M}$, and $\hat{\sigma}^2_{\gamma(\alpha\beta)M}$, respectively. Let us also denote the corresponding scaled ML estimates by $\hat{W}_{\alpha M} = (\hat{\sigma}^2_{\alpha M}/\sigma^2_{\alpha})$, $\hat{W}_{\beta(\alpha)M} = (\hat{\sigma}^2_{\beta(\alpha)M}/\sigma^2_{\beta(\alpha)})$, and $\hat{W}_{\gamma(\alpha\beta)M} = (\hat{\sigma}^2_{\gamma(\alpha\beta)M}/\sigma^2_{\gamma(\alpha\beta)})$. These estimates do not have closed-form expressions, and their exact distributions are unknown. Their quantiles cannot therefore be obtained exactly. For this reason, the QDGs based on these estimates can only be derived empirically using the so-called *empirical quantile dispersion graphs* (EQDGs). The EQDGs for $\hat{W}_{\alpha M}$, $\hat{W}_{\beta(\alpha)M}$, and $\hat{W}_{\gamma(\alpha\beta)M}$ for each design are obtained on the basis of the following Monte-Carlo simulation:

(1) Without any loss of generality, we assume that $\mu = 0$ in model (1). We then have

$$y_{ijkl} = \alpha_i + \beta_{ij} + \gamma_{ijk} + \epsilon_{ijkl}, \tag{13}$$

$i = 1, 2, \ldots, a$, $j = 1, 2, \ldots, b_i$, $k = 1, 2, \ldots, c_{ij}$, and $l = 1, 2, \ldots, n_{ijk}$. Recall that $\eta_1 = (\sigma_\alpha^2/\sigma_\epsilon^2)$, $\eta_2 = (\sigma_{\beta(\alpha)}^2/\sigma_\epsilon^2)$, and $\eta_3 = (\sigma_{\gamma(\alpha\beta)}^2/\sigma_\epsilon^2)$. We can then write $\alpha_i \sim N(0, \eta_1\sigma_\epsilon^2)$, $\beta_{ij} \sim N(0, \eta_2\sigma_\epsilon^2)$, $\gamma_{ijk} \sim N(0, \eta_3\sigma_\epsilon^2)$, and $\epsilon_{ijkl} \sim N(0, \sigma_\epsilon^2)$. Let $R$ denote a region in the parameter space of the quadruple $(\boldsymbol{\eta}, \sigma_\epsilon^2)$, where $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$.

(2) For a given value of $(\boldsymbol{\eta}, \sigma_\epsilon^2)$ in R and a given design D, a random vector $\boldsymbol{y}$ is generated on the basis of model (13): one random number, which we denote by $r_i$, is generated from $N(0, \eta_1\sigma_\epsilon^2)$. Independently from $r_i$, $b_i$ independent random numbers, which we denote by $s_{i1}, s_{i2}, \ldots, s_{ib_i}$, are generated from $N(0, \eta_2\sigma_\epsilon^2)$. Furthermore, $c_{ij}$ independent random numbers, which we denote by $t_{ij1}, t_{ij2}, \ldots, t_{ijc_{ij}}$, are generated from $N(0, \eta_3\sigma_\epsilon^2)$, independently from $r_i$ and $s_{ij}$. Finally, $n_{ijk}$ independent random numbers, which we denote by $u_{ijk1}, u_{ijk2}, \ldots, u_{ijkn_{ijk}}$, are generated from $N(0, \sigma_\epsilon^2)$, independently from $r_i$, $s_{ij}$, and $t_{ijk}$. This process produces the data vector, $(r_i + s_{i1} + t_{i11} + u_{i111}, r_i + s_{i1} + t_{i11} + u_{i112}, \ldots, r_i + s_{ib_i} + t_{ib_ic_{ib_i}} + u_{ib_ic_{ib_i}n_{ib_ic_{ib_i}}})'$, which represents response values from the $i$th first-stage nesting group ($i = 1, 2, \ldots, a$). By repeating this process independently $a$ times and combining the results, we obtain a vector $\boldsymbol{y}$ consisting of $\sum_{i=1}^a \sum_{j=1}^{b_i} \sum_{k=1}^{c_{ij}} n_{ijk} = N$ observations.

(3) Using the response vector $\boldsymbol{y}$ obtained in Step (2), ML estimates, $\hat{\sigma}_{\alpha\mathrm{M}}^2$, $\hat{\sigma}_{\beta(\alpha)\mathrm{M}}^2$, and $\hat{\sigma}_{\gamma(\alpha\beta)\mathrm{M}}^2$ are computed, and values of $\hat{W}_{\alpha\mathrm{M}} = (\hat{\sigma}_{\alpha\mathrm{M}}^2/\eta_1\sigma_\epsilon^2)$, $\hat{W}_{\beta(\alpha)\mathrm{M}} = (\hat{\sigma}_{\beta(\alpha)\mathrm{M}}^2/\eta_2\sigma_\epsilon^2)$, and $\hat{W}_{\gamma(\alpha\beta)\mathrm{M}} = (\hat{\sigma}_{\gamma(\alpha\beta)\mathrm{M}}^2/\eta_3\sigma_\epsilon^2)$ are thus obtained.

(4) By repeating Steps (2) and (3), a sufficient number of times, we can obtain empirical quantiles of $\hat{W}_{\alpha\mathrm{M}}$, $\hat{W}_{\beta(\alpha)\mathrm{M}}$, and $\hat{W}_{\gamma(\alpha\beta)\mathrm{M}}$ for selected values of $p$ for a given $(\boldsymbol{\eta}, \sigma_\epsilon^2)$. We denote the $p-$ th empirical quantile thus obtained by $\tilde{q}_{\mathrm{D}}^{\alpha}(p, \boldsymbol{\eta}, \sigma_\epsilon^2)$ for $\hat{W}_{\alpha\mathrm{M}}$, $\tilde{q}_{\mathrm{D}}^{\beta}(p, \boldsymbol{\eta}, \sigma_\epsilon^2)$ for $\hat{W}_{\beta(\alpha)\mathrm{M}}$, and $\tilde{q}_{\mathrm{D}}^{\gamma}(p, \boldsymbol{\eta}, \sigma_\epsilon^2)$ for $\hat{W}_{\gamma(\alpha\beta)\mathrm{M}}$.

(5) Steps (2)–(4) are repeated several times corresponding to different values of $(\boldsymbol{\eta}, \sigma_\epsilon^2)$ selected from the region R. The set of such selected quadruples is denoted by H. Thus, H $\subset$ R.

(6) The minimum and maximum of the $p$-th empirical quantile values over the set H in Step (5) are obtained for each of $\hat{W}_{\alpha\mathrm{M}}$, $\hat{W}_{\beta(\alpha)\mathrm{M}}$, and $\hat{W}_{\gamma(\alpha\beta)\mathrm{M}}$. We then have for $\hat{W}_{\alpha\mathrm{M}}$,

$$\tilde{Q}_{\mathrm{D},\alpha}^{\min}(p) = \min_{(\boldsymbol{\eta},\sigma_\epsilon^2)\in \mathrm{H}} \tilde{q}_{\mathrm{D}}^{\alpha}(p, \boldsymbol{\eta}, \sigma_\epsilon^2) \quad \text{and} \quad \tilde{Q}_{\mathrm{D},\alpha}^{\max}(p) = \max_{(\boldsymbol{\eta},\sigma_\epsilon^2)\in \mathrm{H}} \tilde{q}_{\mathrm{D}}^{\alpha}(p, \boldsymbol{\eta}, \sigma_\epsilon^2), \tag{14}$$

for $\hat{W}_{\beta(\alpha)\mathrm{M}}$,

$$\tilde{Q}_{\mathrm{D},\beta}^{min}(p) = \min_{(\boldsymbol{\eta},\sigma_\epsilon^2)\in \mathrm{H}} \tilde{q}_{\mathrm{D}}^{\beta}(p, \boldsymbol{\eta}, \sigma_\epsilon^2) \quad \text{and} \quad \tilde{Q}_{\mathrm{D},\beta}^{\max}(p) = \max_{(\boldsymbol{\eta},\sigma_\epsilon^2)\in \mathrm{H}} \tilde{q}_{\mathrm{D}}^{\beta}(p, \boldsymbol{\eta}, \sigma_\epsilon^2), \tag{15}$$

and for $\hat{W}_{\gamma(\alpha\beta)\mathrm{M}}$,

$$\tilde{Q}_{\mathrm{D},\gamma}^{\min}(p) = \min_{(\boldsymbol{\eta},\sigma_\epsilon^2)\in \mathrm{H}} \tilde{q}_{\mathrm{D}}^{\gamma}(p, \boldsymbol{\eta}, \sigma_\epsilon^2) \quad \text{and} \quad \tilde{Q}_{\mathrm{D},\gamma}^{\max}(p) = \max_{(\boldsymbol{\eta},\sigma_\epsilon^2)\in \mathrm{H}} \tilde{q}_{\mathrm{D}}^{\gamma}(p, \boldsymbol{\eta}, \sigma_\epsilon^2). \tag{16}$$

(7) The minimum and maximum quantile values obtained from Equations (14) to (16) are plotted against $p$ to obtain the EQDGs of $\hat{W}_{\alpha\mathrm{M}}$, $\hat{W}_{\beta(\alpha)\mathrm{M}}$, and $\hat{W}_{\gamma(\alpha\beta)\mathrm{M}}$, respectively.

Table 2. Minimum and maximum quantile values of $\hat{W}_{\alpha \mathrm{M}}$, $\hat{W}_{\beta(\alpha)\mathrm{M}}$, and $\hat{W}_{\gamma(\alpha\beta)\mathrm{M}}$ for selected values of $p$ for designs $D_{\mathrm{bal}}$, $D_{\mathrm{stg}}$, and $D_{\mathrm{inv}}$.

| | $\hat{W}_{\alpha M}$ | | | | | | $\hat{W}_{\beta(\alpha)M}$ | | | | | | $\hat{W}_{\gamma(\alpha\beta)\mathrm{M}}$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $D_{\mathrm{bal}}$ | | $D_{\mathrm{stg}}$ | | $D_{\mathrm{inv}}$ | | $D_{\mathrm{bal}}$ | | $D_{\mathrm{stg}}$ | | $D_{\mathrm{inv}}$ | | $D_{\mathrm{bal}}$ | | $D_{\mathrm{stg}}$ | | $D_{\mathrm{inv}}$ | |
| $p$ | Min. | Max. | Min. | Max. | Min. | Max. | Min. | Max. | Min. | Max. | Min. | Max. | Min. | Max. | Min. | Max. | Min. | Max. |
| 0.01 | 0.000 | 0.031 | 0.000 | 0.153 | 0.000 | 0.239 | 0.000 | 0.048 | 0.000 | 0.153 | 0.000 | 0.169 | 0.000 | 0.252 | 0.000 | 0.236 | 0.000 | 0.157 |
| 0.05 | 0.000 | 0.170 | 0.000 | 0.319 | 0.000 | 0.401 | 0.000 | 0.185 | 0.000 | 0.307 | 0.000 | 0.350 | 0.000 | 0.397 | 0.000 | 0.380 | 0.000 | 0.308 |
| 0.10 | 0.000 | 0.277 | 0.000 | 0.420 | 0.000 | 0.499 | 0.000 | 0.287 | 0.000 | 0.421 | 0.000 | 0.451 | 0.143 | 0.485 | 0.000 | 0.473 | 0.000 | 0.419 |
| 0.20 | 0.000 | 0.445 | 0.000 | 0.576 | 0.000 | 0.642 | 0.000 | 0.444 | 0.000 | 0.572 | 0.000 | 0.604 | 0.358 | 0.616 | 0.230 | 0.627 | 0.000 | 0.577 |
| 0.30 | 0.000 | 0.579 | 0.000 | 0.705 | 0.000 | 0.765 | 0.000 | 0.583 | 0.000 | 0.698 | 0.000 | 0.726 | 0.532 | 0.738 | 0.441 | 0.752 | 0.298 | 0.700 |
| 0.40 | 0.000 | 0.746 | 0.000 | 0.821 | 0.000 | 0.865 | 0.000 | 0.729 | 0.000 | 0.826 | 0.000 | 0.851 | 0.689 | 0.855 | 0.638 | 0.866 | 0.556 | 0.819 |
| 0.50 | 0.280 | 1.002 | 0.342 | 1.113 | 0.312 | 1.128 | 0.000 | 0.891 | 0.123 | 1.058 | 0.173 | 1.192 | 0.831 | 0.966 | 0.825 | 0.969 | 0.801 | 0.996 |
| 0.60 | 0.924 | 2.368 | 0.980 | 2.316 | 1.012 | 2.327 | 0.631 | 1.596 | 0.915 | 2.256 | 0.948 | 2.476 | 0.935 | 1.131 | 0.934 | 1.178 | 0.909 | 1.246 |
| 0.70 | 1.181 | 4.015 | 1.149 | 3.661 | 1.164 | 3.580 | 1.013 | 2.702 | 1.057 | 3.534 | 1.065 | 3.736 | 1.043 | 1.335 | 1.041 | 1.421 | 1.029 | 1.573 |
| 0.80 | 1.441 | 5.979 | 1.363 | 5.442 | 1.317 | 5.283 | 1.211 | 4.158 | 1.205 | 5.036 | 1.188 | 5.402 | 1.158 | 1.594 | 1.164 | 1.752 | 1.158 | 1.979 |
| 0.90 | 1.827 | 9.214 | 1.641 | 8.028 | 1.534 | 7.551 | 1.520 | 6.365 | 1.420 | 7.232 | 1.368 | 7.675 | 1.356 | 2.008 | 1.339 | 2.258 | 1.335 | 2.589 |
| 0.95 | 2.254 | 12.540 | 1.889 | 10.115 | 1.730 | 9.805 | 1.789 | 8.552 | 1.613 | 9.141 | 1.530 | 9.689 | 1.546 | 2.386 | 1.495 | 2.698 | 1.459 | 3.127 |
| 0.99 | 2.989 | 19.395 | 2.350 | 14.585 | 2.119 | 14.162 | 2.332 | 13.071 | 1.980 | 13.589 | 1.829 | 13.649 | 1.866 | 3.210 | 1.804 | 3.733 | 1.734 | 4.549 |

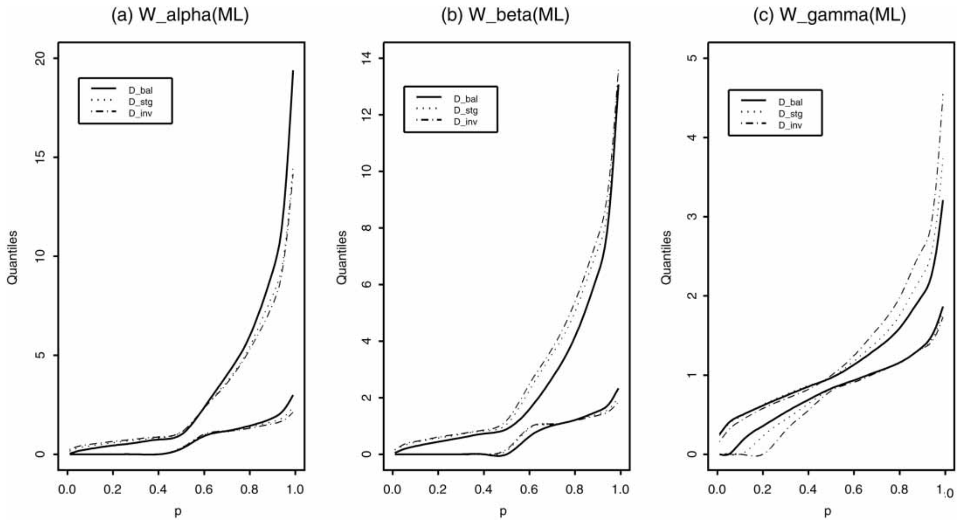


Figure 2. Empirical quantile dispersion graphs of (a) $\hat{W}_{\alpha M} = \hat{\sigma}^2_{\alpha M}/\sigma^2_\alpha$, (b) $\hat{W}_{\beta(\alpha)M} = \hat{\sigma}^2_{\beta(\alpha)M}/\sigma^2_{\beta(\alpha)}$, and (c) $\hat{W}_{\gamma(\alpha\beta)M} = \hat{\sigma}^2_{\gamma(\alpha\beta)M}/\sigma^2_{\gamma(\alpha\beta)}$ for designs, $D_{bal}$, $D_{stg}$, and $D_{inv}$, where $\hat{\sigma}^2_{\alpha M}$, $\hat{\sigma}^2_{\beta(\alpha)M}$, and $\hat{\sigma}^2_{\gamma(\alpha\beta)M}$ are the maximum likelihood estimates of the variance components.

To apply the above procedure for the designs $D_{bal}$, $D_{stg}$, and $D_{inv}$ in Leone et al. [16], we chose the region R as

$$\mathrm{R} = \left\{(\boldsymbol{\eta}, \sigma^2_\epsilon) \mid 1 \le \eta_1 \le 9,\ 1 \le \eta_2 \le 9,\ 1 \le \eta_3 \le 9,\ 0.1 \le \sigma^2_\epsilon \le 10\right\}. \tag{17}$$

The set H mentioned in Step (5) consists of 3645 points selected from R such that

$$\mathrm{H} = \{(\boldsymbol{\eta}, \sigma^2_\epsilon) \mid \eta_1 = 1(1)9,\ \ \eta_2 = 1(1)9,\ \ \eta_3 = 1(1)9,\ \ \sigma^2_\epsilon = 0.1, 0.5, 1, 5, 10\}.$$

A SAS Macro program was written to compute $\tilde{q}^\alpha_D(p, \boldsymbol{\eta}, \sigma^2_\epsilon)$, $\tilde{q}^\beta_D(p, \boldsymbol{\eta}, \sigma^2_\epsilon)$, and $\tilde{q}^\gamma_D(p, \boldsymbol{\eta}, \sigma^2_\epsilon)$, and PROC MIXED in SAS [25] was used to obtain the ML estimates of the variance components. A total of 2000 replications in Step (3) were made to calculate empirical quantile values for each design as well as for each quadruple, $(\boldsymbol{\eta}, \sigma^2_\epsilon)$, in the region H. Table 2 gives values of the minima and maxima as defined in Equations (14) through (16) for designs $D_{bal}$, $D_{stg}$, and $D_{inv}$.

The EQDGs of $\hat{W}_{\alpha M}$, $\hat{W}_{\beta(\alpha)M}$, and $\hat{W}_{\gamma(\alpha\beta)M}$ are given in Figure 2. Similar results to those of ANOVA estimators are observed: designs $D_{stg}$ and $D_{inv}$ provide more consistent ML estimates of $\sigma^2_\alpha$ than $D_{bal}$ (Figure 2a). For $\sigma^2_{\beta(\alpha)}$ and $\sigma^2_{\gamma(\alpha\beta)}$, however, the superiority of $D_{bal}$ to $D_{stg}$ and $D_{inv}$ is apparent (Figure 2b and c). Design $D_{stg}$ is preferred to design $D_{inv}$ when the ML estimation of $\sigma^2_{\gamma(\alpha\beta)}$ is considered, otherwise, there is no clear distinction between them.

## 5. Comparisons of ANOVA and ML estimation

Comparisons of ANOVA estimation with that of ML with regard to the variance components are made in Figures 3–5. Aside from the possibility of having negative estimates of the variance components, the ANOVA estimates are comparable to those of ML with respect to $\sigma^2_\alpha$ and $\sigma^2_{\beta(\alpha)}$ (Figures 3 and 4). This holds true for all the three designs. The variability in the quantile values with respect to $\boldsymbol{\eta}$ for $\hat{W}_{\gamma(\alpha\beta)A}$ is even smaller than that for $\hat{W}_{\gamma(\alpha\beta)M}$ with respect to $(\boldsymbol{\eta}, \sigma^2_\epsilon)$, except for the possibility of getting negative values of the variance component estimate (Figure 5). This holds true for all the three designs and over the entire range of $p$. The probability of negative

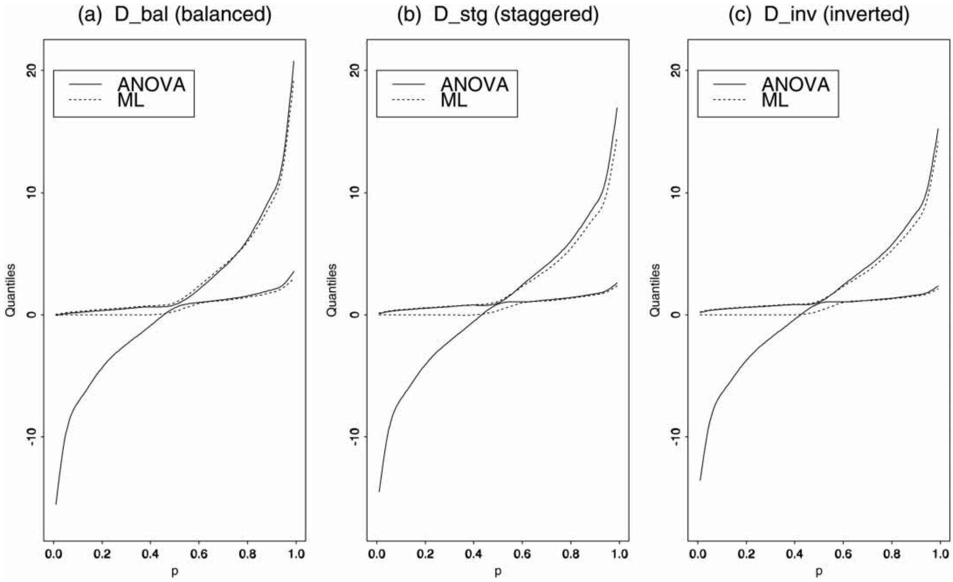


Figure 3. Quantile dispersion graphs of $\hat{W}_{\alpha\mathrm{A}}$ and empirical quantile dispersion graphs of $\hat{W}_{\alpha\mathrm{M}}$ for designs $\mathrm{D_{bal}}$, $\mathrm{D_{stg}}$, and $\mathrm{D_{inv}}$.

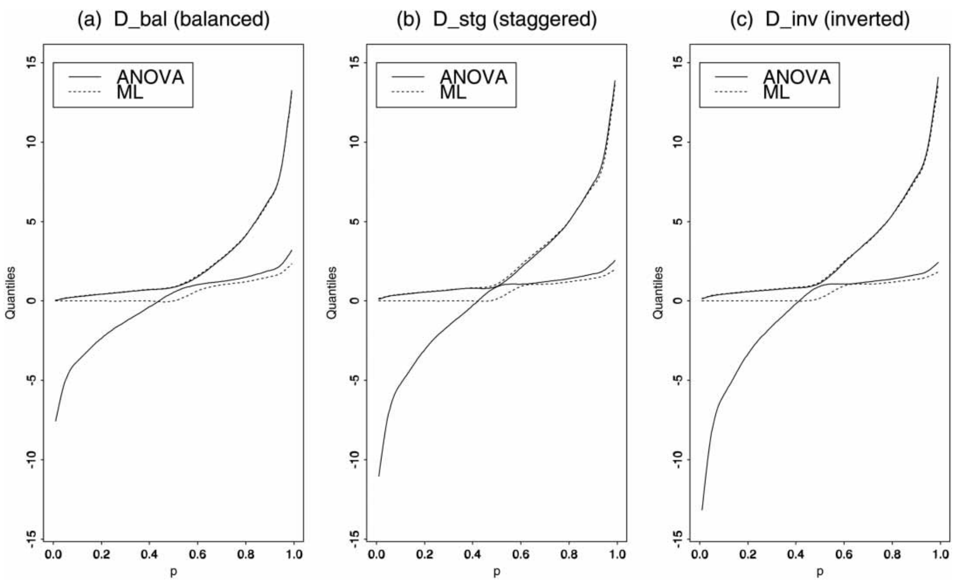


Figure 4. Quantile dispersion graphs of $\hat{W}_{\beta(\alpha)\mathrm{A}}$ and empirical quantile dispersion graphs of $\hat{W}_{\beta(\alpha)\mathrm{M}}$ for designs $\mathrm{D_{bal}}$, $\mathrm{D_{stg}}$, and $\mathrm{D_{inv}}$.

ANOVA estimates for $\sigma_\alpha^2$ and $\sigma_{\beta(\alpha)}^2$ is about 40% at the very most for all designs, while that of $\sigma_{\gamma(\alpha\beta)}^2$ is relatively small for the balanced and staggered designs. It should, however, be mentioned that this comparison is limited since the regions S in Section 3.2 and R in Equation (17) are not identical.

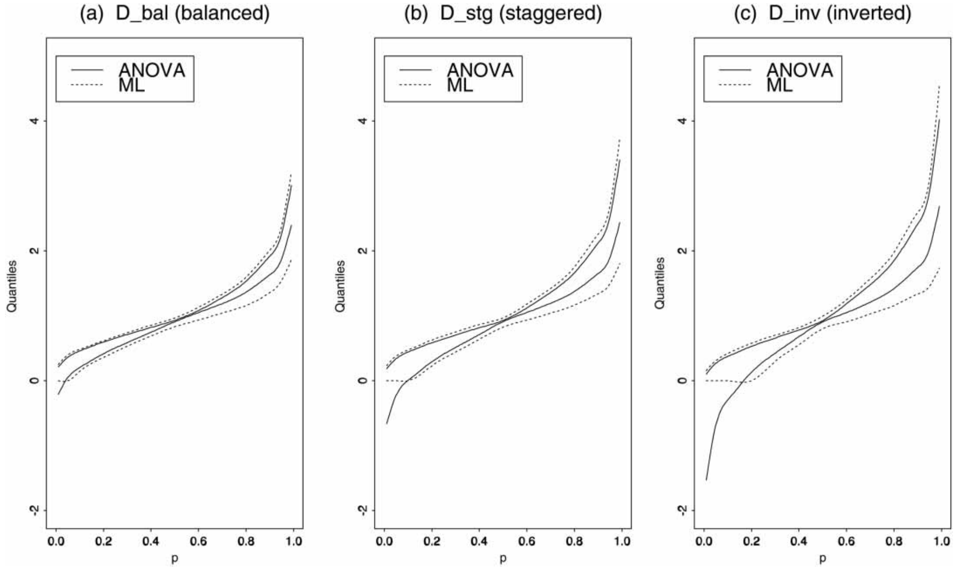


Figure 5. Quantile dispersion graphs of $\hat{W}_{\gamma(\alpha\beta)\mathrm{A}}$ and empirical quantile dispersion graphs of $\hat{W}_{\gamma(\alpha\beta)\mathrm{M}}$ for designs $\mathrm{D}_{\mathrm{bal}}$, $\mathrm{D}_{\mathrm{stg}}$, and $\mathrm{D}_{\mathrm{inv}}$.

## 6. Conclusion

In this study, three designs for the three-fold nested random model, namely, the balanced, staggered, and inverted nested designs, are compared using a graphical approach, the so-called QDGs. Two estimation methods, namely ANOVA and ML, are considered. The QDGs allow us to compare the overall quality of estimation of the variance components for the three designs.

Both the staggered and the inverted nested designs are almost as efficient as the balanced nested design in terms of consistency and stability in estimating the variance component of the highest nesting factor. With other nested factors, however, the staggered nested design provides more variable estimates than the balanced design, while the inverted design produces the largest variability, especially for the lowest nested factor.

A comparison of the ANOVA estimation with that of ML is also made. Even though the comparison is limited by the fact that the two regions in the corresponding parameter spaces considered are not identical, it was demonstrated that ANOVA estimation provides comparable estimates to those of ML, except for the possibility of having negative ANOVA estimates. This is an interesting result because it is based on a small sample property of ML estimation, rather than on its asymptotic behavior.

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