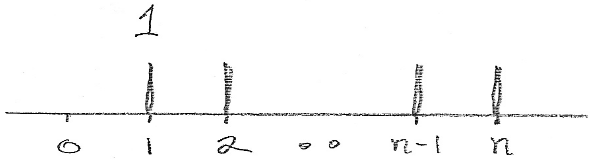
Chapter 5 - Annuities

Section 5.3 - Review of Annuities-Certain

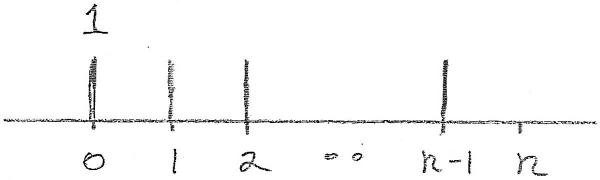
Annuity Immediate - It pays 1 at the end of every year for n years.



The present value of these payments is:

where $\nu = \frac{1}{1+i}$.

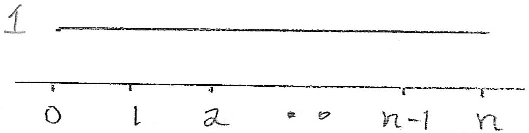
Annuity-Due - It pays 1 at the beginnig of every year for n years.



The present value of these payments is:

where $d = \frac{i}{1+i}$.

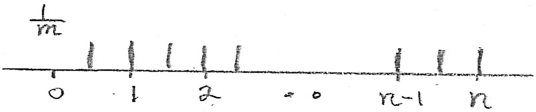
Continuous Payment Annuity - It smears the payment of 1 over each year for n years.



The present value of this smear of payments is:

where $\delta = \ln(1 + i)$.

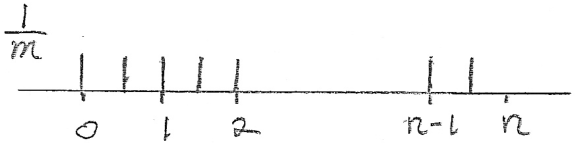
$m^{th}ly$ Annuity Immediate - It pays $\frac{1}{m}$ at the end of every $\frac{1}{m}$ part of the year for n years.



The present value of these payments is:

where $\left(1 + \frac{i(m)}{m}\right)^m = (1 + i) = \nu^{-1}$.

m^{th}*ly* Annuity Due - It pays $\frac{1}{m}$ at the beginning of every $\frac{1}{m}$ part of the year for n years.



The present value of these payments is:

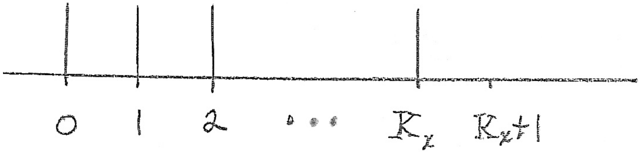
where $\left(1 - \frac{d(m)}{m}\right)^{-m} = (1 + i) = \nu^{-1}$.

Section 5.4 - Annual Life Annuities

The annual life annuity pays the annuitant (annuity policyholder) once each year as long as the annuitant is alive on the payment date. If the policy continues to pay throughout the remainder of the annuitant's life, it is called a whole life annuity.

Subsection 5.4.1 - Whole Life Annuity-Due

Payments of \$1 are made at the beginning of each year of the annuitant's remaining life. The present value random variable is



The only random part of this expression is ν^{K_x+1}. We have already found in chapter 4 (pages 4-6, 4-7) that

$$E\left[\nu^{K_x+1}\right] = A_x \qquad \text{and}$$

$$E\left[\left(\nu^{K_x+1}\right)^2\right] = {}^2A_x.$$

Because Y is a linear function of ν^{K_x+1}, we immediately get the EPV and Var[Y] to be

An alternative expression for the EPV can be found by noting that

$$Y = \mathrm{I}(\mathrm{T_x} > 0) + \nu \mathrm{I}(\mathrm{T_x} > 1) + \nu^2 \mathrm{I}(\mathrm{T_x} > 2) + \cdots,$$

where $\mathrm{I}(\{\text{event}\}) = 1$ if the event occurs and 0 otherwise.

Clearly,

We also note that

$$Y = (1)\mathrm{I}(0 \le \mathrm{T_x} < 1) + (1+\nu)\mathrm{I}(1 \le \mathrm{T_x} < 2)$$
$$+(1+\nu+\nu^2)\mathrm{I}(2 \le \mathrm{T_x} < 3) + \cdots \qquad \text{or}$$

$$Y = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \, \mathrm{I}(\mathrm{k} \leq \mathrm{T_x} < \mathrm{k} + 1).$$

This produces

A whole life annuity-due could be used to describe annual payments from an insurance company to an individual under a lifetime annuity contract. It can also be used to describe the annual premiums paid by an individual to the insurance company which is used to fund the individual's life insurance.

Subsection 5.4.2 - Term Annuity-Due

The present value random variable of $1 annual payments under an annuity due contract with a maximum of n payments is:

$$Y = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x < n \\ \ddot{a}_{\overline{n}|} & \text{if } K_x \geq n \end{cases}$$

$$= \frac{1 - \nu^{min(K_x+1,n)}}{d}.$$

It follows that the EPV is

Note that $A_{x:\overline{n}|}$ is the endowment insurance EPV. See page 4-21.

This EPV can also be written as

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \nu^k \, {}_k p_x \qquad \text{or}$$

The variance of Y can be written as

$$Var[Y] = \left[\sum_{k=0}^{n-1} (\ddot{a}_{\overline{k+1}|})^2 \left({}_k | q_x \right) \right] + (\ddot{a}_{\overline{n}|})^2 \left({}_n p_x \right) - \left(\ddot{a}_{x:\overline{n}|} \right)^2 .$$

A term annuity-due is often used to describe the annual premiums paid by an individual to the insurance company which is used to fund the individual's life insurance in settings which have a maximum number of premium payments.

Subsection 5.4.3 - Whole Life Annuity-Immediate

In this setting \$1 annual payments are made at the end of each year provided the annuitant is alive at that point in time.

Denote the present value random variable of these payments by

$$Y^* = \nu \mathrm{I}(\mathrm{T_x} > 1) + \nu^2 \mathrm{I}(\mathrm{T_x} > 2) + \cdots,$$

Comparing this expression to Y described at the top of page 5-6, shows that

$$Y^* = Y - 1.$$

Therefore, the expected value and the variance satisfy

Subsection 5.4.4 - Term Annuity-Immediate

In this setting the \$1 annual payments are made at the end of each year provided the annuitant is alive at that point in time, but there will be at most n payments made. The present value random variable is

$$Y_n^* = \nu \mathrm{I}(\mathrm{T_x} \geq 1) + \nu^2 \mathrm{I}(\mathrm{T_x} \geq 2) + \cdots + \nu^{\mathrm{n}} \mathrm{I}(\mathrm{T_x} \geq \mathrm{n}),$$

$$= a_{\overline{\min(K_x,n)}|} = \frac{1 - \nu^{\min(K_x,n)}}{i}.$$

It follows that its EPV is

Comparing this to

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \nu^k \left({}_k p_x\right),$$

we see that

Since

$$Y_n^* = \left[\sum_{k=1}^{n-1} a_{\overline{k}|} \, \mathrm{I}(\mathrm{k} \le \mathrm{T_x} < \mathrm{k} + 1) \right] + a_{\overline{n}|} \mathrm{I}(\mathrm{T_x} \ge \mathrm{n}),$$

the variance of Y_n^* can be written as

$$Var[Y_n^*] = \left[\sum_{k=1}^{n-1} (a_{\overline{k}|})^2 \left({}_{k|}q_x \right) \right] + (a_{\overline{n}|})^2 \left({}_n p_x \right) - \left(a_{x:\overline{n}|} \right)^2.$$

Example 5-1: You are given ${}_{10}p_0 = .07$, ${}_{20}p_0 = .06$ and ${}_{30}p_0 = .04$. Suppose each survivor age 20 contributes P to a fund so there is an amount at the end of 10 years to pay \$1,000 to each survivor age 30. Use $i = .06$ and find P.

Example 5-2: You are given (1) 10 year pure endowment of 1, (2) whole life annuity-immediate with 1 annual payments, (3) whole life annuity-due with 1 annual payments and (4) 10-year temporary life annuity-immediate with 1 for annual payments. Rank the actuarial present values of these options.

Example 5-3: An insurance company agrees to make payments to someone age x who was injured at work. The payments are \$150K annually, starting immediately and continuing as long as the person is alive. After the first \$500K the remainder is paid by a reinsurance company. Let $i = .05$, ${}_tp_x = (.7)^t$ for $0 \leq t \leq 5.5$ and 0 for $t > 5.5$. Calculate the EPV for the reinsurance company.

Example 5-4: You are given: $P[T_0 > 25] = .7$ and $P[T_0 > 35] = .5$.
Each of the following annuities-due have an actuarial PV of 60,000:
(1) life annuity-due of 7,500 on (25)
(2) life annuity-due of 12,300 on (35)
(3) life annuity-due of 9,400 on (25) that makes at most 10 payments
What is the interest rate?

Section 5.5 - Annuities Payable Continuously

Subsection 5.5.1 - Whole Life Continuous Annuity

In this setting a continuous payment of \$1 is smeared over each year until time t (not necessarily an integer). The present value of this smear is

$$\bar{a}_{\overline{t|}} = \int_0^t \nu^s ds = \frac{\nu^s}{\ln(\nu)}\Big|_0^t = \frac{\nu^t - 1}{\ln(\nu)} = \frac{1-\nu^t}{\delta}$$

So when the life length T_x is random, the present value random variable is

$$Y = \bar{a}_{\overline{T_x|}} = \frac{1-\nu^{T_x}}{\delta}.$$

Its present value is

and its variance is

Recall that $1 - F_x(t) = {}_tp_x$ and $f_x(t) = -\frac{d}{dt}\,{}_tp_x$. Also that

$$\frac{d}{dt}\overline{a}_{\overline{t}|} = \frac{d}{dt}\left(\frac{1-\nu^t}{-\ln(\nu)}\right) = \nu^t$$

Therefore,

$$\overline{a}_x = E\big[\overline{a}_{\overline{T_x}|}\big] = \int_0^\infty \overline{a}_{\overline{t}|} f_x(t)dt = \int_0^\infty \overline{a}_{\overline{t}|}\big(-\frac{d}{dt}\,{}_tp_x\big)dt$$

$$= -(\overline{a}_{\overline{t}|})\,{}_tp_x\Big|_0^\infty + \int_0^\infty \nu^t\,{}_tp_x dt. \qquad \text{int by parts}$$

It follows that

Subsection 5.5.2 - Term Life Continuous Annuity

The present value random variable for a term continuous annuity setting ending at n (not necessarily an integer) years is

$$Y = \bar{a}_{\overline{min(T_x, n)}|} = \frac{1 - \nu^{min(T_x, n)}}{\delta},$$

So its EPV is

and its variance is

$$Var[Y] = \frac{Var\left[\nu^{min(T_x, n)}\right]}{\delta^2} = \frac{{}^2\bar{A}_{x:\overline{n}|} - \left(\bar{A}_{x:\overline{n}|}\right)^2}{\delta^2}.$$

Recall that $\bar{A}_{x:\overline{n}|}$ is the endowment EPV for life insurance. See pages 4-20 and 4-21.

Other expressions for the EPV include

$$\bar{a}_{x:\overline{n}|} = \int_0^n e^{-\delta t} {}_t p_x dt$$

and

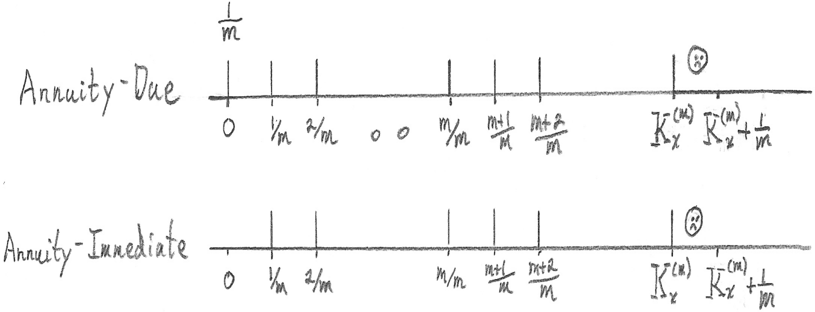
$$\bar{a}_{x:\overline{n}|} = \int_0^n (\bar{a}_{\overline{t}|}) {}_t p_x \mu_{x+t} dt + (\bar{a}_{\overline{n}|}) {}_n p_x.$$

Example 5-5: Important Setting Assume a constant force of mortality μ_x^* and a constant force of interest δ. Find $\overline{a}_x$ and $\overline{a}_{x:\overline{n}|}$.

Example 5-6: Important Setting Assume a constant force of interest δ and a future life length that is de Moivre $(0, \omega - x)$. Find $\overline{a}_x$ and $\overline{a}_{x:\overline{n}|}$.

Section 5.6 - Annuities Payable m^{th}ly

Subsection 5.5.1 - Whole Life Annuity Payable m^{th}ly



Here $K_x^{(m)} = \dfrac{\text{(\# of sub-periods survived)}}{m}$ and the payments are each $\frac{1}{m}$ dollars, so the total payment over each year is \$1.

For a whole life annuity-due setting, the present value random variable is

$$Y = \ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}|} = \frac{1-\nu^{(K_x^{(m)}+\frac{1}{m})}}{d^{(m)}}.$$

The EPV is then

and the variance is

$$Var[Y] = \frac{Var\left[\nu^{(K_x^{(m)}+\frac{1}{m})}\right]}{(d^{(m)})^2} = \frac{{}^2A_x^{(m)} - \left(A_x^{(m)}\right)^2}{(d^{(m)})^2}$$

An alternative form for this EPV is

$$\ddot{a}_x^{(m)} = \sum_{k=0}^{\infty} \left(\frac{1}{m}\right) \nu^{\frac{k}{m}} \left({}_{\frac{k}{m}}p_x \right).$$

For a whole life annuity-immediate, we see that

Because their present value random variables differ only by a constant, the variance expression for an annuity-immediate is identical to the one for the annuity-due.

Subsection 5.6.2 - Term Life Annuity Payable m^{th}ly

When payments cease after n years, the present value random variable of the annuity-due is:

$$Y = \ddot{a}^{(m)}_{\overline{\min(K_x^{(m)}+\frac{1}{m},n)}|} = \frac{1 - \nu^{\min(K_x^{(m)}+\frac{1}{m},n)}}{d^{(m)}}.$$

So its EPV is:

and its variance is

$$Var[Y] = \frac{Var\left[\nu^{\min(K_x^{(m)}+\frac{1}{m},n)}\right]}{(d^{(m)})^2} = \frac{{}^2A^{(m)}_{x:\overline{n}|} - \left(A^{(m)}_{x:\overline{n}|}\right)^2}{(d^{(m)})^2}$$

where $A^{(m)}_{x:\overline{n}|}$ denotes an endowment EPV.

An alternative form for the EPV is

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \sum_{k=0}^{mn-1} \left(\frac{1}{m}\right) \nu^{\frac{k}{m}} \left({}_{\frac{k}{m}}p_x\right).$$

We also note that the term annuity-immediate EPV, $a^{(m)}_{x:\overline{n}|}$, is just like $\ddot{a}^{(m)}_{x:\overline{n}|}$ except that it does not contain the constant first term $\frac{1}{m}$ at $k = 0$ and does contain a term at $k = mn$, that is

$$a^{(m)}_{x:\overline{n}|} = \sum_{k=1}^{mn} \left(\frac{1}{m}\right) \nu^{\frac{k}{m}} \left({}_{\frac{k}{m}}p_x\right).$$

In an annual annuity-due, a payment of 1 occurs at the beginning of the year. When m payments are of $\frac{1}{m}$ are made during the year, they move the mass (potential payments) further away from $t = 0$. Thus,

$$\ddot{a}_x > \ddot{a}_x^{(m)} \qquad \text{for } m > 1.$$

Similar reasoning applied to an annuity-immediate shows that

$$a_x < a_x^{(m)} \qquad \text{for } m > 1.$$

It follows that when $m > 1$,

The same relationships hold when comparing term annuity EPV's.

Example 5-7: Mortality follows de Moivre (0,80) for someone age x. If $\delta = .05$, find $\overline{a}_{x:\overline{20}|}$.

Example 5-8: If Y is the present value R.V. for someone age x, find $E[Y]$ when

$$Y = \bar{a}_{\overline{n}|} \quad \text{when } 0 \leq T_x \leq n \quad \text{and} \quad Y = \bar{a}_{\overline{T_x}|} \quad \text{otherwise.}$$

Example 5-9: You are give that $\mu = .01$ and that $\delta = .04$ when $0 \leq t \leq 5$ and $\delta = .03$ when $t > 5$. Find $\overline{a}_{x:\overline{10}|}$.

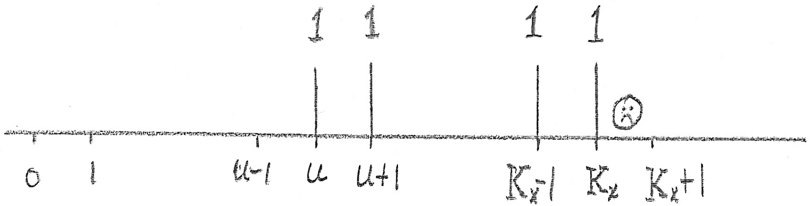
Example 5-10: Given $\delta = .05$ and $\mu_t = .05$ for $0 \le t \le 50$ and $\mu_t = .10$ for $t > 50$. Find the EPV of a 50 year temporary life continuous payment annuity with 1 smeared over each year for insured age 30, ceasing immediately at death.

Example 5-11: A 30 year old will receive \$5,000 annually beginning one year from today for as long as this person lives. When $p_x = k$ for all x, the actuarial PV of the annuity is \$22,500, with $i = .10$. Find k.

Example 5-12: Given $A_x = .28$, $A_{x+20} = .40$, $A_{x:\overline{20}|}^{\;\;1} = .25$ and $i = .05$, find $a_{x:\overline{20}|}$.

Section 5.8 - Deferred Life Annuities

A deferred life annuity does not begin payments until u years have passed and then continues to make annual payments of 1 as long as the annuitant is alive at the time of payment.



Here K_x is the number of whole years survived by the annuitant, i.e. $K_x = \lfloor T_x \rfloor$. Also the distinction between an annuity-due and an annuity-immediate is moot in this setting. We could use either, but choose to use annuity-due.

Let

$$Y_x = (\text{ 1 paid at the beginning of each year through } K_x) \quad \text{and}$$

$$Y_{x:\overline{u}|} = (\text{ 1 paid at the beginning of each year through } K_x,$$
$$\text{with a maximum of } u \text{ payments}).$$

The present value random variable for the deferred life annuity would then be

$${}_{u|}Y_x = Y_x - Y_{x:\overline{u}|}.$$

It follows that the EPV is

A second way to view a deferred whole life annuity is to consider its payments year by potential year:

$$
\begin{aligned}
{}_{u|}\ddot{a}_x &= E[\,{}_{u|}Y_x\,] \\
&= 1\,{}_{u}p_x \nu^u + 1\,{}_{u+1}p_x \nu^{u+1} + 1\,{}_{u+2}p_x \nu^{u+2} + \cdots \\
&= {}_{u}p_x \nu^u \left(1 + 1\,{}_{1}p_{x+u}\nu^1 + 1\,{}_{2}p_{x+u}\nu^2 + \cdots\right) \\
&= {}_{u}p_x\, \nu^u\, \ddot{a}_{x+u}
\end{aligned}
$$

Once again we see ${}_{u}E_x$ acting like a discount function. Similarly, when the annuity is a term (max of n payments) annuity,

and for m^{th}ly payment annuities

$$
{}_{u|}\ddot{a}_x^{(m)} = {}_{u}E_x \left(\ddot{a}_{x+u}^{(m)}\right).
$$

Example 5-13: A continuous deferred annuity pays at a rate of $(1.04)^t$ at time t starting 5 years from now. You are given $\delta = .07$ and $\mu_t = .01$ when $t < 5$ and $\mu_t = .02$ when $t \geq 5$. Find the actuarial present value of this annuity.

Section 5.9 - Guaranteed Annuities

Money accumulated in a 401K account is typically used to purchase a retirement annuity which will pay the annuitant a monthly income for the remainder of the annuitant's life. These life annuities come with guarantee options of 0, 5, 10, 15, or 20 years. The longer the guarantee period, the lower the monthly payment amounts, but these decrements are usually not substantial.

Consider an annual whole life annuity with a n-year guarantee period, for which the annuitant is x years old. The present value random variable of $1 annual payments is

$$Y = \begin{cases} \ddot{a}_{\overline{n}|} & \text{if } K_x \leq n-1 \\ \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x \geq n \end{cases}$$

where ${}_{n|}Y_x$ is the present value of a n-year deferred annual annuity-due.

Thus the EPV of this n-year guaranteed annuity is

$$\ddot{a}_{\overline{x:\overline{n}|}} = E[\,Y\,] = \ddot{a}_{\overline{n}|} + E[\,_{n}|Y_x\,] \qquad \text{or}$$

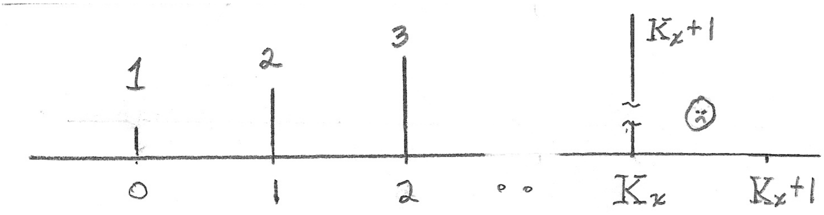
When payments are made m^{th}ly, the EPV becomes

Example 5-14: Given $\overline{A}_x = .3$, $\overline{A}_{x:\overline{20|}} = .4$, $i = .05$ and ${}_{20}p_x = .7$, find $\overline{a}_{\overline{x:\overline{20|}}}$.

Example 5-15: An annuity pays 1 at the beginning of each year for (35) and pays until age 65. Payments are certain for the first 15 years. Calculate the EPV given: $\ddot{a}_{\overline{15|}} = 11.94$, $\ddot{a}_{\overline{30|}} = 19.6$, $\ddot{a}_{35:\overline{15|}} = 11.62$, $\ddot{a}_{35:\overline{30|}} = 18.13$, $\ddot{a}_{35} = 21.02$, $\ddot{a}_{50} = 15.66$ and $\ddot{a}_{65} = 9.65$.

Section 5.10 - Life Annuities With Increasing Payments

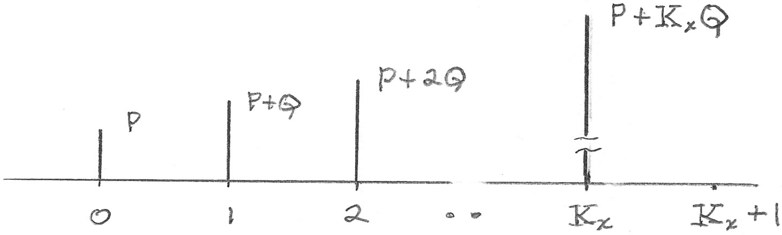
Subsection 5.10.1 - Arithmetically Increasing Annuities



Consider the arithmetically increasing sequence of life contingent payments pictured above. The EPV of these payments is

If it is a term policy with n as the maximum number of annual payments, then the EPV is

In general, suppose the arithmetically increasing sequence of payments is as shown below.



The EPV in this general case is

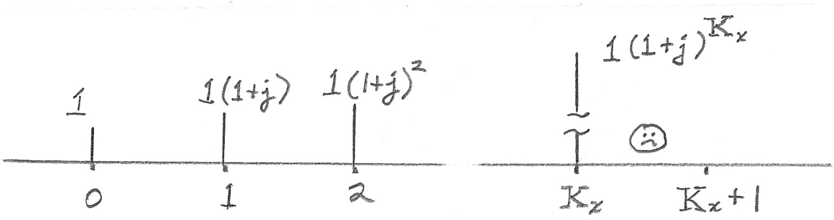
$$(P)\,\ddot{a}_x \;+\; (Q)\,{}_1E_x\,(\mathrm{I}\ddot{a})_{x+1}\,.$$

With a continuous payment of t at time t, the EPV becomes

If the payments are a linear function of time, for example, $P_t = bt + c$, where b and c are constants, then the EPV is

$$b\left(\bar{\mathrm{I}}\bar{a}\right)_x \;+\; c\bar{a}_x.$$

Subsection 5.10.2 - Geometrically Increasing Annuities



Consider a sequence of life contingent annuity payments that increase geometrically as shown above. The EPV of this sequence of payments is

$$\sum_{t=0}^{\infty}(1+j)^t \nu^t {}_tp_x = \sum_{t=0}^{\infty}\left(\frac{1+j}{1+i}\right)^t {}_tp_x = \sum_{t=0}^{\infty}\left(\frac{1}{1+i^*}\right)^t {}_tp_x$$

$$= \ddot{a}_{x\, i^*}$$

Here i^* is the interest rate that satisfies

Example 5-16: An injured worker is to receive annual annuity payments beginning today with a payment of \$100K. Subsequent payments increase by 2% each year. If $i = .05$ and ${}_tp_x = (.7)^t$ for $0 \leq t \leq 5$ and ${}_tp_x = 0$ for $t > 5$, find the EPV.

Example 5-17: A person age (20) buys a special 5-year life contingent annuity-due with annual payments of 1, 3, 5, 7 and 9. Find the EPV if

$$\ddot{a}_{20:\overline{4}|} = 3.41 \quad a_{20:\overline{4}|} = 3.04 \quad (\mathrm{I}\ddot{a})_{20:\overline{4}|} = 8.05 \quad (\mathrm{I}a)_{20:\overline{4}|} = 7.17$$

Example 5-18: Age (65) considers three term life contingent annuities paying annually with last payment at age 75.

1) \$5K at age 66, subsequent payments decrease by \$500 per year, has EPV of \$14K

2)\$1K at age 65, subsequent payments increase by \$1K per year, has EPV of \$21K

3) \$1K every year with first at (65), has EPV of P

Find P.

Section 5.11 - Evaluating Life Annuities

Subsection 5.11.1 - Recursive Evaluation for Life Annuities

Values for $\ddot{a}_x$ are very useful in assessing a lifetime annuity or a potential stream of life contingent payments. Recall that

$$\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_nE_x \ddot{a}_{x+n}$$

so the values for $\ddot{a}_x$ can be used to find term EPV's , $\ddot{a}_{x:\overline{n}|}$, also. The whole life values are often computed recursively, starting at the end with

$$\ddot{a}_{\omega-1} = 1,$$

and working back over time by using

$$\begin{aligned} \ddot{a}_x &= 1 + \nu p_x + \nu^2 p_x p_{x+1} + \nu^3 p_x p_{x+1} p_{x+2} + \cdots \\ &= 1 + \nu p_x \left(1 + \nu p_{x+1} + \nu^2 p_{x+1} p_{x+2} + \cdots\right) \qquad \text{or} \end{aligned}$$

Similarly, for an m^{th}ly annuity-due:

Subsection 5.11.2 - Applying the UDD Assumption

Under the UDD assumption concerning death within each year, the values of $\ddot{a}_x^{(m)}$ and $\ddot{a}_{x:\overline{n}|}^{(m)}$ can be computed from either A_x or $\ddot{a}_x$ values. Recall that under UDD,

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x \qquad \text{and} \qquad \overline{A}_x = \frac{i}{\delta} A_x.$$

This produces

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}} = \frac{1 - \frac{i}{i^{(m)}} A_x}{d^{(m)}} \qquad \text{or}$$

Now use the fact that $\ddot{a}_x = \frac{1-A_x}{d}$ and write

$$\ddot{a}_x^{(m)} = \frac{i^{(m)} - i(1 - d\ddot{a}_x)}{i^{(m)} d^{(m)}} = \left(\frac{id}{i^{(m)} d^{(m)}} \right) \ddot{a}_x - \left(\frac{i - i^{(m)}}{i^{(m)} d^{(m)}} \right) \quad \text{or}$$

As $m \to \infty$, we get

because $\lim_{m\to\infty} i^{(m)} = \delta = \lim_{m\to\infty} d^{(m)}$.

Likewise, for term annuity-dues,

$$\begin{aligned}\ddot{a}^{(m)}_{x:\overline{n}|} &= \ddot{a}^{(m)}_{x} - {}_nE_x \ddot{a}^{(m)}_{x+n} \\ &= \alpha(m)\ddot{a}_x - \beta(m) - {}_nE_x \left\{\alpha(m)\ddot{a}_{x+n} - \beta(m)\right\} \\ &= \alpha(m)\left\{\ddot{a}_x - {}_nE_x\ddot{a}_{x+n}\right\} - \beta(m)\left(1 - {}_nE_x\right) \qquad \text{or}\end{aligned}$$

Note also that

$$\alpha(m) \approx 1 \qquad \text{and} \qquad \beta(m) \approx \frac{m-1}{2m}.$$

(See exercise 5.15 the the textbook.)

Subsection 5.11.3 - Woolhouse Approximations

Consider the function

$$g(t) \equiv \nu^t {}_tp_x = e^{-\delta t} {}_tp_x.$$

Note that

$$g'(t) = -\delta e^{-\delta t} {}_tp_x - e^{-\delta t} {}_tp_x \mu_{x+t}.$$

It follows that

$$g(0) = 1 \qquad \lim_{t\to\infty} g(t) = 0 \quad \text{and} \quad g'(0) = -(\delta + \mu_x).$$

A result from numerical integration based on the Euler-Maclaurin expansion shows that for $h > 0$, Formula:

$$\int_0^\infty g(t)dt = h\sum_{k=0}^{\infty} g(kh) - \frac{h}{2}g(0) + \frac{h^2}{12}g'(0) + \frac{h^4}{720}g''(0) + \cdots$$

The approximation will ignore $g''(0)$ and higher order derivative terms.

(1) Applying this formula with $h = 1$ yields

$$\int_0^\infty g(t)dt \doteq \sum_{k=0}^{\infty} g(k) - \frac{1}{2}g(0) + \frac{1}{12}g'(0)$$

$$= \sum_{k=0}^{\infty} \nu^k {}_kp_x - \frac{1}{2} - \frac{1}{12}(\delta + \mu_x)$$

$$= \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\delta + \mu_x).$$

(2) Applying this formula with $h = \frac{1}{m}$ yields

$$\int_0^\infty g(t)dt \doteq \frac{1}{m}\sum_{k=0}^{\infty} g(\frac{k}{m}) - \frac{1}{2m}g(0) + \frac{1}{12m^2}g'(0)$$

$$= \frac{1}{m}\sum_{k=0}^{\infty} \nu^{\frac{k}{m}}\,{}_{\frac{k}{m}}p_x - \frac{1}{2m} - \frac{1}{12m^2}(\delta + \mu_x)$$

$$= \ddot{a}_x^{(m)} - \frac{1}{2m} - \frac{1}{12m^2}(\delta + \mu_x).$$

(3) Setting these two approximations equal to each other produces

$$\ddot{a}_x^{(m)} - \frac{1}{2m} - \frac{1}{12m^2}(\delta + \mu_x) \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\delta + \mu_x) \qquad \text{or}$$

Letting $m \to \infty$, we get for continuously paying annuities that

$$\bar{a}_x \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\delta + \mu_x)$$

The corresponding approximation for term annuities is:

$$\ddot{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \left(\frac{m-1}{2m}\right)(1 - {}_nE_x) - \left(\frac{m^2-1}{12m^2}\right)\left\{(\delta + \mu_x) - {}_nE_x(\delta + \mu_{x+n})\right\}.$$

The Woolhouse approximations above require the use of μ_x. With life table data, we can approximate μ_x as follows:

$$ {}_2p_{x-1} = p_x p_{x-1} = e^{-\int_{x-1}^{x+1} \mu_s ds} \approx e^{-2\mu_x}. $$

This motivates

where $p_x = \frac{l_{x+1}}{l_x}$.

Section 5.13 - Select Lives in Life Annuities

Selection via underwriting alters the survival probabilities which in tern alter the present values of payment streams. For example,

$$\ddot{a}_x = \sum_{k=0}^{\infty} \nu^k \, {}_k p_{[x]}$$