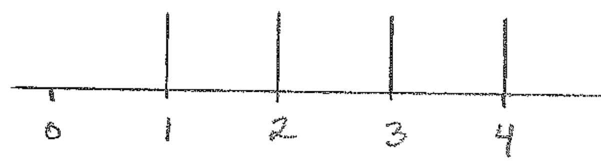
Chapter D8 Swaps

A forward contract involves the purchase of an asset at a fixed predetermined date. In practice companies frequently need to purchase commodities periodically not just once. So they seek to enter one agreement that will cover all of these periodic needs.

Swap – An agreement involving the periodic exchange of payments over time. It provides a hedge to a periodic stream of risk exposures and implicitly involves the borrowing and lending of money.

Commodity Swap

Example: Suppose Company Q needs one million gallons of gasoline delivered every year for four years beginning one year from today. They are thus



exposed to substantial risk due to the price variation of gasoline. They could hedge this risk by entering into 4 forward contracts (one for year 1, one for year 2, etc.) Suppose the price of the forward for the j^{th} year and the corresponding interest rate of a zero coupon bond expiring in j years are:

Years Ahead	Forward Contract Price / Gallon	Zero Coupon interest rate
1	\$3.72	5%
2	\$3.93	5.5%
3	\$4.20	6%
4	\$4.51	6.25%

So were the company to invest

$$\frac{3.72}{(1.05)} + \frac{3.93}{(1.055)^2} + \frac{4.20}{(1.06)^3} + \frac{4.51}{(1.0625)^4} = 14.1390$$
(million)

total into four zero-coupon bonds, they would have the cash available to pay for the four forward contracts. Or paying 14.1390 today purchases the four forwards (a <u>prepaid swap</u>)

Instead of the above scheme, the company agrees to one <u>swap</u> contract obligating them to pay x dollars/gallon for 1M gallons <u>each</u> of the next four years from today. The value x would be chosen so that

$$\frac{x}{(1.05)} + \frac{x}{(1.055)^2} + \frac{x}{(1.06)^3} + \frac{x}{(1.0625)^4} = 14.1390$$

or

$$x = 4.068639.$$

The swap buyer agrees to pay this

amount per gallon at the end of each year of the four years for 1 M gallons each year. Compared to the forward price, the buyer is <u>overpaying</u> by:

at year 1 $4.068639 - 3.72 = .348639$

at year 2 $4.068639 - 3.93 = .138639$

and then <u>underpaying</u> by:

at year 3 $\boxed{4.2 - 4.068639 = .131361}$

at year 4 $\boxed{4.51 - 4.068639 = .441361}$ per gallon.

So the buyer is loaning money to the swap counterpary during the first two years and being repaid in years 3 and 4. The interest rate at which these cash flows balance is the **implied interest rate** of the swap. So select a time t (for example, the end of year 4) and compute it, producing:

$$(.348639)(1+i)^3 + (.138639)(1+i)^2$$
$$= (.131361)(1+i) + (.441361).$$

Setting $x = 1+i$, yields

$$(.348639)x^3 + (.138639)x^2 - (.131361)x - .441361 = 0$$

or

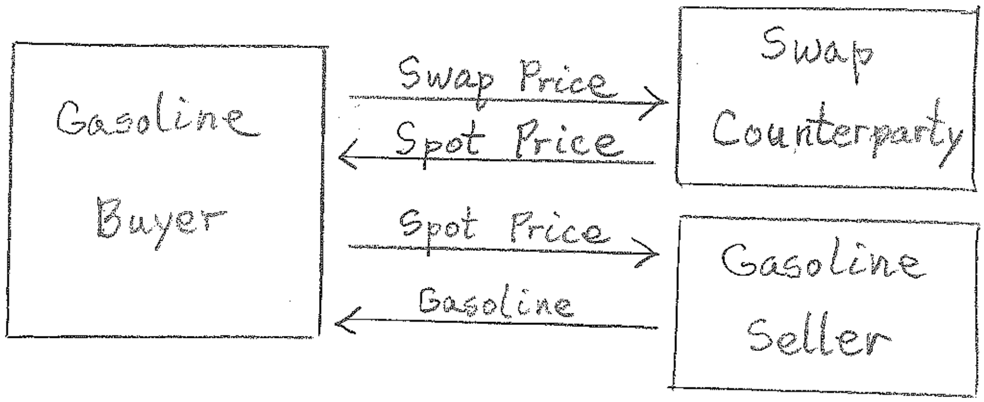
$$x^3 + (.397658)x^2 - (.376782)x - 1.26595 = 0.$$

Solving this cubic equation, we see that

$x = 1.0671233$ or $i = .0671233$

is the only viable solution. Thus 6.71233 % is the implied interest rate.

A swap is typically arranged by a swap counterparty (a market maker) who finds another seller of the swap or hedges its position in the swap with other agreements.



Most often when the spot price and the swap price differ, there is merely a financial settlement between the buyer and the swap counterparty, as indicated in the above diagram, for the difference between the swap and spot prices.

In general, let

F_{0,t_j} = forward contract price of the purchase needed at time t_j

$P(0,t_j)$ = price of a zero coupon bond expiry at time t_j from today with a redemption value of 1

then

$$\text{price of prepaid swap} = \boxed{\sum_{j=1}^{n} \left(F_{0,t_j}\right) P(0,t_j)}$$

and the fixed swap price R must satisfy

$$\boxed{\sum_{j=1}^{n} R\,P(0,t_j) = \sum_{j=1}^{n}\left(F_{0,t_j}\right)P(0,t_j).}$$

Therefore,

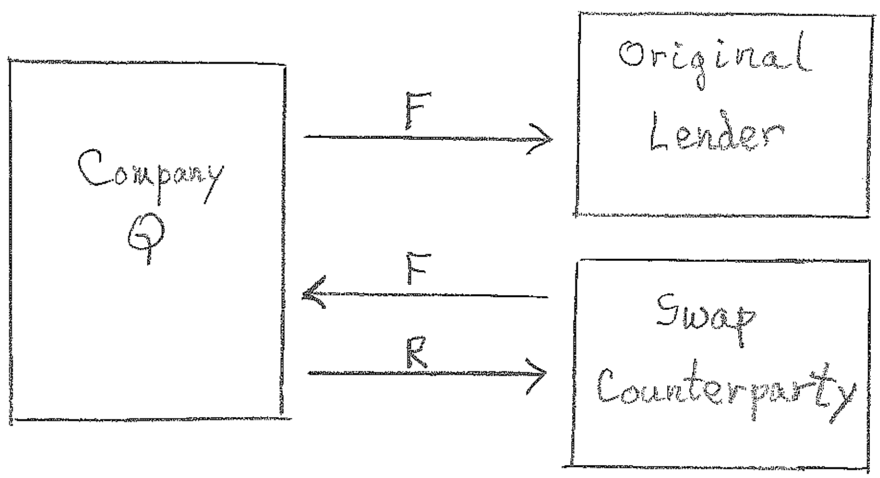
$$R = \frac{\sum_{j=1}^{n}\left(F_{0,t_j}\right)P(0,t_j)}{\sum_{j'=1}^{n} P(0,t_{j'})}$$

$$= \boxed{\sum_{j=1}^{n}\left[\frac{P(0,t_j)}{\sum_{j'=1}^{n} P(0,t_{j'})}\right]F_{0,t_j}}$$

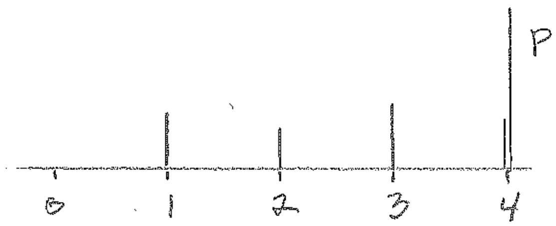
which is just a weighted average of the forward contract prices with weights (adding to one) proportional to the corresponding zero-coupon bond price.

Interest Rate Swaps

Example: Suppose company Q has a 10 million dollar loan with a floating interest rate (LIBOR+1%) over a four year period. If interest rates are volatile, this exposes the company to substantile risk if rates should rise. They may seek an agreement with a swap counterparty to swap their floating rate F for a fixed interest rate R.



In this agreement the counterparty assumes the risk associated with the floating rate F and received an appropriate (?) fixed rate R from company Q.



We view the company's debt situation as a series of simple interest payments at the ends of years 1, 2, 3 and 4 plus the return of the principal at the end of year 4. The principal is known, but the interest payments are not due to the floating interest rate.

Company Q could enter into a "strip" of forward rate agreements that that would pay the difference between the floating rate appropriate at the end of a particular year and an agreed upon fixed rate. Suppose in this setting:

t years from today	$r_0(0,t)$ Zero coupon bond yield
1	.055
2	.060
3	.065
4	.068

Thus a fixed interest rate associated with year 1 is $r_0(0,1) = .055$. For the year between $t=1$ and $t=2$, the appropriate rate viewed from today ($t=0$) is the implied forward rate $r_0(1,2)$ satifying

$$(1+.055)(1+r_0(1,2)) = (1+r_0(0,2))^2$$
$$= (1+.060)^2$$

or

$$\boxed{r_0(1,2) = \frac{(1+.060)^2}{(1+.055)} - 1 = .0650237}$$

In general, for the year that begins at time $t-1$ and ends at time t the appropriate rate as viewed from today (time 0) is the one-year implied forward rate $r_0(t-1,t)$ that satisfies

$$(1+r_0(0,t-1))^{t-1}(1+r_0(t-1,t)) = (1+r_0(0,t))^t$$

or

$$r_0(t-1,t) = \boxed{\frac{(1+r_0(0,t))^t}{(1+r_0(0,t-1))^{t-1}}} - 1 .$$

See page 10-16 at the bottom. This process produces a new set of values

t years from today	$r_0(0,t)$ Zero Coupon bond yield	$r_0(t-1,t)$ One-Year Implied Forward Rate
1	.055	.055
2	.060	.0650237
3	.065	.0750709
4	.068	.0770508

The present value of this strip of forward rate agreements is

$$PV = \frac{.055}{1.055} + \frac{.0650237}{(1.06)^2} + \frac{.0750709}{(1.065)^3}$$

$$+ \frac{.0770508}{(1.068)^4}$$

$$= .231374176$$

price of a stip of prepaid forward rate agreements

But we desire a constant future rate agreement at rate R. That would require its present value to satisfy

$$\boxed{\frac{R}{(1.055)} + \frac{R}{(1.060)^2} + \frac{R}{(1.065)^3} + \frac{R}{(1.068)^4} = .231374176}$$

or

$$R = \frac{.231374176}{3.4343387} = .06737$$

which becomes the interest rate of the fixed rate swap, ie the <u>swap rate</u>.

In general, if

$r_0(0,t) =$ Zero coupon bond yield for a bond maturing at time t

$r_0(t-1,t) =$ One year implied forward rate for year $t-1$ to t, then the <u>swap rate</u> is:

$$R = \frac{\displaystyle\sum_{t=1}^{n} \frac{r_0(t-1,t)}{(1+r_0(0,t))^t}}{\displaystyle\sum_{t'=1}^{n} \frac{1}{(1+r_0(0,t'))^{t'}}} .$$

Since

$$P(0,t) \overset{\text{def}}{\equiv} \frac{1}{(1+r_0(0,t))^t}$$

is the price of a zero-coupon bond maturing at a value of 1 at time t, we can <u>write</u>

$$\boxed{R = \sum_{t=1}^{n} \left[\frac{P(0,t)}{\displaystyle\sum_{t'=1}^{n} P(0,t')} \right] r_0(t-1,t)}$$

and see it as the weighted average of the one-year implied forward interest rates with weights (adding to 1) proportional to the price of the corresponding zero coupon bond.

The $\boxed{\text{swap spread}}$ is the difference between the swap rate and a U.S. Treasury bond yield with the same maturity.

A $\boxed{\text{deferred swap}}$ is one that is agreed to and priced today but does not start until a specified time in the future. If in the earlier example, the swap was set to involve only the interest payments at end of years 3 and 4, then

$$R = \frac{(1.065)^{-3}(.0750709) + (1.068)^{-4}(.0770508)}{\left[(1.065)^{-3} + (1.068)^{-4}\right]}$$

$$= .076024 \quad .$$

In general,

$$R = \frac{\sum_{t=k}^{n} P(0,t)\, r_0(t-1,t)}{\sum_{s=k}^{n} P(0,s)} ,$$

when the swap begins for the k^{th} interest period.

Exercise D8-A

What are the relations among

(a) $r_0(0,n)$ = (Spot rate for year n)
= (Interest rate of zero-coupon bond maturing at $t=n$)

(b) $r_0(n-1,n)$ = (One-year implied forward interest rate)

(c) $P(0,n)$ = (Price of a zero-coupon bond maturing for \$1 at time $t=n$)

Exercise D8-B Suppose the spot interest rates are 4%, 4.5% and 5.1% for a zero-coupon bond maturing in one, two or three years, respectively. Find the level interest rate for a three-year interest rate swap.

Exercise D8-1 In each case below (they are not related) you are given one set of values for either $r_0(0,n)$ or $r_0(n-1,n)$ or $P(0,n)$. In each case find the other two sets for $n = 1, 2, 3$.

n	Case (a) $r_0(0,n)$	Case (b) $r_0(n-1,n)$	Case (c) $P(0,n)$
1	.055	.042	.96712
2	.062	.058	.91398
3	.078	.064	.85892

Three unrelated problem settings

Exercise D8-2 The forward prices for forward contracts on a particular stock are shown to the right. Using the rates of Case (a) in exercise D8-1, find the level price of a 3-year swap contract on this stock.

n	$F_{0,n}$
1	125
2	130
3	138

Exercise D8-3 Using the results of Case (c) in exercise D8-1, find the interest rate of a level 3-year interest rate swap.

Derivative Chapters Exercise Answers

D2-1 20

D2-2 $20+5=25$

D2-3 False / True / True

D2-4 73.48

D2-5 No / No / Yes

D2-6 −4

D2-7 103.64

D2-8 −2.96

D2-9 113

D3-1 Buy 1 @ 10, Sell 1 @ 25, Sell 1 @ 45, Buy 1 @ 60

D3-2 (c) Buy 1 call @ 100, Sell 2 calls @ 125, Buy 2 calls @ 175

D3-3 520.23

D3-4 Sell a Straddle

D3-5 Buy a collar

D4-1 None

D4-2 (a) 1.16 (b) −7.98 (c) −4.33 Spread = 9.14

D5-1 102.92

D5-2 80.74

D5-3 92.28

D5-4 $195,654.75

D8-1 (c) $r_0(0,n)$: .034 .046 .052

$r_0(n-1,n)$: .034 .05814 .06410

D8-2 130.63

D8-3 .05149