Chapter D5 – Forwards and Futures

Suppose we desire to own a particular stock at a future time $t = T > 0$, where $t = 0$ denotes today. Let S_0 be the current price of this stock and S_T be its unknown future value (price) at time T. There are four basic methods we could use today to ensure we own the stock at time T:

Method of Buying Stock	Time of Payment	Time Stock is Received	Amount of Payment
Outright purchase	0	0	S_0
Borrow to pay for stock ("Fully leveraged")	T	0	S_0e^{rT}
Prepaid forward contract	0	T	
Forward contract	T	T	

The first option is to pay S_0 and purchase the stock today. The second option is to purchase it today, but to borrow the purchase price S_0, agreeing to repay it with interest at time T. Here r is the continuous force of interest attached to the loan.

<u>Forward Contract</u> —

<u>Prepaid Forward Contract</u> - The asset (current value S_0) is paid for today but is not delivered until time $T > 0$ when its value is S_T (unknown).

Pricing a Prepaid Forward Contract (with no dividends):

This price should be the discounted expected value of the asset when we receive it at time T, i.e. $e^{-rT} E[S_T]$ where r is an interest rate appropriate for this time. But we estimate $E[S_T]$ with

$$S_0 e^{rT}.$$

(a) So the appropriate price is:

$$e^{-rT} E[S_T] \quad \text{estimated by}$$

(b) We now argue that this is the correct price under the "No Arbitrage Principle" (in a competitive market there should be no investments with a guaranteed positive cash flow).

Let $F_{0,T}^{P}$ = Price of Prepaid Forward Contract

If $\underline{F_{0,T}^{P} > S_0}$ then

Buy the stock at S_0 and sell a prepaid forward at $F_{0,T}^{P}$ produces

Gain of $(F_{0,T}^{P} - S_0)$ right now and you have the asset to deliver at time T.

If $\underline{S_0 > F^P_{0,T}}$ then

Buy the prepaid forward now at $F^P_{0,T}$ and Borrow the stock now and sell it at S_0. You immediately gain $(S_0 - F^P_{0,T})$ plus you will be able to replace the borrowed stock at time T with no risk.

Therefore, if there is to be no arbitrage, it must be that

Pricing Prepaid Forward Contracts (with dividends)

Suppose dividend D_{t_j} are paid at time t_j. Since we are buying the asset today, but deferring possession until time T (each $t_j < T$), the dividends which we should have but will not receive (appropriately discounted) are deducted from the price we pay, i.e. the price should be

when n dividends are paid prior to ownership transfer.

When the asset is a portfolio of stocks, for example, dividends are frequent and the growth in the fund due to dividends would be approximated by

$$\left(1 + \frac{\delta}{365}\right)^{365T} \doteq e^{\delta T}$$

so we view δ as the annualized continuous rate of dividend payment. In this setting the price of the prepaid forward contract should be:

$$F^P_{0,T} =$$

In this setting of continuous growth we can buy $e^{-\delta T}$ units of the asset now and by reinvesting the dividends, (which continuously compound) we would have

$$(e^{-\delta T})e^{\delta T} = 1$$

unit at time T. This process is described as a ________________ in the asset.

So the prepaid forward contract price

$$F_{0,T}^{P} =$$

is a ______ version of the outright purchase price. By reinvesting the dividends you would have one full unit of value

$$S_T$$

at time T.

Pricing Forward Contracts $F_{0,T}$

With no dividends, the price should be:

$$E[S_T] \text{ estimated with } e^{rT} S_0$$

where r is again the appropriate force of interest. We see this is just the future value of the price of a prepaid forward contract, i.e.

$$F_{0,T} = e^{rT} F_{0,T}^{P}.$$

So when discrete dividends are applied at times t_j,

$$F_{0,T} = e^{rT}\left(S_0 - \sum_{j=1}^{n} e^{-rt_j} D_{t_j}\right)$$

$$=$$

and for a portfolio with continuous dividend growth:

$$F_{0,T}$$

Synthetic Forward (Role of the Market Maker)

Consider a setting in which a client agrees to a forward contract on a stock index. The client agrees to pay $F_{0,T}$ at time T and also at time T the client will receive the stock index value of S_T.

The broker (market maker) who makes this deal happen must cover its position, i.e. it must have $S_T - F_{0,T}$ available at time T. The market maker makes money on commissions, etc. and is not particularly speculative. So what does the market maker do to cover its position?

The market maker <u>buys</u> a tailed position in the index for at time $t=0$. It borrows the money to make this purchase at the risk-free rate of r. By reinvesting the dividends of the $S_0 e^{-\delta T}$ shares, it grows the account to one full share with value S_T at time T. But at time T it must repay the loan, i.e. it must pay

$$F^P_{0,T}\, e^{rt} = S_0\, e^{(r-\delta)T}.$$

So at time T, the market maker has assets of

which is just what it needs to cover its position in the original forward contract in which the client is said to be in the long position (buyer) and the market maker is in the short position (seller).

Note that the MM does not speculate on the value of S_T. So it is risk-free and is equivalent to the purchase of a risk-free zero coupon bond at price $S_0 e^{-\delta T}$ with redemption value $F_{0,T}$. The interest rate attached to this bond, ie r satisfying

$$F_{0,T} =$$

is called the ______________ .

The difference between the risk-free rate r and the dividend yield δ, ie $(r-\delta)$ is called the cost of carry, and it represents the net cost of financing the position in the asset.

Cash and Carry – Buy an asset and take the short position in a forward contract on this asset – it is risk free.

Forward Price vs Future Spot Price

$F_{0,T}$ = Forward Price

S_T = Future Spot Price at T

Let $T = 1$, one year.

Let α be the annual expected return on the asset, i.e.

$$E[S_1] = S_0(1+\alpha).$$

Let r be the risk-free annual rate of return on investments. Then

$$F_{0,1} = S_0(1+r).$$

The forward price is thus a biased estimator of the future spot price.

$$E[S_1] - F_{0,1} = (\alpha - r)S_0.$$

Here $(\alpha - r)$ is the risk premium, the compensation for bearing the risk of the investment.

Typically α is unknown and hence so is $(\alpha - r)$. If we buy the asset at $T=0$ and borrow the full amount at rate r, we get

$$S_0(1+\alpha) - S_0(1+r) = (\alpha - r)S_0,$$

a risk premium return at no cost to us. This is a similar position to the forward contract on the previous page.

The forward price formula is:

$$F_{0,T} = S_0 e^{(r-\delta)T}$$

where r is the risk-free interest rate and δ is the annualized daily compound dividend rate. The quantity $(r-\delta)$ is described as the <u>cost of carry rate</u>.

To explain this terminology, again assume $T=1$. Suppose you buy an asset at S_0 and borrow to fund this purchase at rate r. At $T=1$, you will owe rS_0 on this loan. But along the way you will receive δS_0 in dividend payments. Thus, the <u>net cost of carrying the long position</u> in this asset is $(r-\delta)S_0$.

Note also that if you take the short position in this asset (i.e. borrow the asset to return it at $T=1$), you must return the asset and you must also reimburse the lender for the dividends paid while you had the asset, i.e. S_0e^{δ}. Thus δ is described as a <u>lease rate</u>.

Exercise D5-A Suppose one share of a mutual fund has current value of \$1,200. The annual rate of fund growth due to dividends has force of interest 2.5% and the risk-free force of interest is 4%.

(a) What should you pay today to purchase one share of this fund one year from today?

(b) If you are to receive one share, one year from today and pay for it one year from today, what price should be agreed to today to be paid in one year?

Exercise D5-B The current price of a stock is \$90. A one-year forward contract is made. The stock pays semi-annual dividends of \$4, one 6-months from today and the other just prior to the exercise date. The risk-free rate of interest is 4% compounded semi-annually. What is the price of a prepaid one-year forward contract?

Futures Contracts

We describe this type of agreement in terms of an index, specifically the S&P 500 index.

The Long position in an S&P 500 index agrees to buy K shares of the index at a fixed price and at a fixed time in the future. (This sounds like a forward contract - BUT)

(a) Futures contracts have set expiration dates (often quarterly).

(b)

(c) A forward contract is settled at the expiration date. A futures contract is settled daily. So as the price varies so does level of your investment.

This daily settlement is called

____

(d) When entering into a futures contract the buyer must give the broker a deposit (called the ______) to ensure no credit risks. Daily interest is earned on this deposit, BUT the deposit level can go up or down depending on the daily settlement of the index. If the margin shrinks to too low of a level the broker makes a margin call requiring more margin to continue the agreement.

(e) The agreement is liquid and thus a buy agreement can be cancelled by a sell agreement, but only the current margin is returned.

Example (McDonald text):

FIGURE 5.1		
Specifications for the S&P 500 index futures contract.	Underlying	S&P 500 index
	Where traded	Chicago Mercantile Exchange
	Size	\$250 × S&P 500 index
	Months	Mar, Jun, Sep, Dec
	Trading ends	Business day prior to determination of settlement price
	Settlement	Cash-settled, based upon opening price of S&P 500 on third Friday of expiration month

The dollar value (notional value) of 8 futures contracts on the S&P 500 is:

$$\underbrace{8 \times \$250}_{\text{multiplier}} \times (\text{current futures price})$$

So if the current index price is 1100, its notional value is

$$2000 \times 1100 = \$2,200,000.$$

Assume that the futures settles weekly and the current annualized force of interest rate is 6%. Assume also that a 10% margin is required by the broker. Table 5.8 shows the weekly margin balance as the balance varies corresponding to the changes in the index futures price.

TABLE 5.8 Mark-to-market proceeds and margin balance over 10 weeks from long position in 8 S&P 500 futures contracts. The last column does not include additional margin payments. The final row represents expiration of the contract.

Week	Multiplier ($)	Futures Price	Price Change	Margin Balance($)
0	2000.00	1100.00	—	220,000.00
1	2000.00	1027.99	−72.01	76,233.99
2	2000.00	1037.88	9.89	96,102.01
3	2000.00	1073.23	35.35	166,912.96
4	2000.00	1048.78	−24.45	118,205.66
5	2000.00	1090.32	41.54	201,422.13
6	2000.00	1106.94	16.62	234,894.67
7	2000.00	1110.98	4.04	243,245.86
8	2000.00	1024.74	−86.24	71,046.69
9	2000.00	1007.30	−17.44	36,248.72
10	2000.00	1011.65	4.35	44,990.57

So, for example, in week 1, the price dropped 72.01. The buyer at 1100 lost

because of this price drop. But gained one week's interest of the previous margin balance

$$\$220{,}000 \times \left(e^{.06/52} - 1\right) = \$253.99.$$

So the margin balance at the end of week 1 is:

$$\begin{array}{r} \$220{,}000.00 \\ +\ 253.99 \\ -\ 144{,}020.00 \\ \hline 76{,}233.99 \end{array}$$

Similarly, during week 2 the index increased 9.89 and

$$(\$76{,}233.99)\, e^{.06/52} + 2000\,(9.89)$$

$$= \$96{,}102.01$$

is thus the balance at the end of week 2. So the margin balance changes weekly as the index price changes. The net profit of the investment over the ten week period is

$$= -177{,}562.59$$

If we had taken a forward contract when the index was 1100 and it fell to 1011.65 over 10 weeks, these same prices would have produced a loss of

The net effect of settling accounts frequently is to magnify loss in this case. A future contract with a fixed expiration in some ways resembles a forward contract. There is a method of treating a future that, in effect, equates the performance of a future and a forward contract. This would enable us to assess the value of the future in comparison to the forward, recognizing the future has a price dictated by exchange-trading and daily settlement. The method involves describing the future as though it is a forward purchase of a fraction of the final shares, where this fraction grows over the length of the investment.

Example:

TABLE 5.13 Marking-to-market proceeds and margin balance from long position in the S&P 500 futures contract, where hedge is adjusted on a weekly basis.

Week	Multiplier ($)	Futures Price	Price Change	Margin Balance ($)
0	1979.34	1100.00	—	217,727.21
1	1981.62	1027.99	−72.01	75,446.43
2	1983.91	1037.88	9.89	95,131.79
3	1986.20	1073.23	35.35	165,372.88
4	1988.49	1048.78	−24.45	117,001.17
5	1990.79	1090.32	41.54	199,738.33
6	1993.09	1106.94	16.62	233,055.86
7	1995.39	1110.98	4.04	241,377.01
8	1997.69	1024.74	−86.24	69,573.25
9	2000.00	1007.30	−17.44	34,813.80
10	2000.00	1011.65	4.35	43,553.99

This table shows the future contract performance of

$$\$250 \times (8.0) e^{-\left(\frac{.06}{52}\right)9} = (\$250)(7.91753) = 1979.34$$

units which grow each week, equaling

$$\$250 \times (8.0)\, e^{-\left(\frac{.06}{52}\right)(n-1-k)}$$

at week k $(k=0,\dots,n-1)$. While the balance changes weekly, the net profit is

$$= -\$176,700$$

exactly that of a forward contract.

Exercise D5-C You purchase a 6-month futures contract on 100 shares of a stock index. Your deposit is \$25,000 in a margin account, which is credited with interest at 5% annual effective. The account is settled weekly. The table below shows the futures price at the end of each week. Find the margin account balance at the end of three weeks.

Week	Futures Price
0	1000
1	1002
2	995
3	998

Why does this work?

$n =$ (total # of settlement periods)

$J =$ (initial multiplier for a forward)

$\alpha =$ (interest rate per settlement period)

$MB_k =$ (margin balance after k^{th} settlement)

$p_k =$ (price of the index at the end of the k^{th} settlement period.)

Note:

$$MB_0 = J p_0 e^{-(n-1)\alpha},$$

$$\text{Multiplier}_k = J e^{-\alpha(n-1-k)}$$

and

$$\underbrace{MB_k}_{\substack{\text{current} \\ \text{period balance}}} = \underbrace{J e^{-(n-k)\alpha}}_{\substack{\text{previous week} \\ \text{multiplier}}} \underbrace{(p_k - p_{k-1})}_{\substack{\text{price index} \\ \text{change}}} + \underbrace{e^{\alpha}}_{\substack{\text{interest} \\ \text{increase}}} \underbrace{MB_k}_{\substack{\text{previous} \\ \text{period balance}}} \qquad *$$

Apply $*$ recursively to the margin balance at the last (n^{th}) settlement period.

$$MB_n = J e^{0} (p_n - p_{n-1}) + e^{\alpha} MB_{n-1}$$

$$= J(p_n - \cancel{p_{n-1}}) +$$

$$e^{\alpha} J e^{-\alpha} (\cancel{p_{n-1}} - p_{n-2}) + e^{2\alpha} MB_{n-2}$$

$$= J p_n - \cancel{J p_{n-2}}$$

$$+ e^{2\alpha} J e^{-2\alpha} (\cancel{p_{n-2}} - p_{n-3}) + e^{3\alpha} MB_{n-3}$$

$$\vdots$$

$$= J p_n - J p_0 + e^{n\alpha} MB_0$$

$$= J(p_n - p_0) + J p_0 e^{\alpha}$$

The net profit of this investment is

$$MB_n - MB_0 e^{n\alpha} =$$

$$J(p_n - p_0) + J p_0 e^{\alpha} - e^{n\alpha} J e^{-(n-1)\alpha} p_0$$

$$= J(p_n - p_0)$$ forward contract value

This <u>does not depend</u> on the intermediate prices $p_1, \ldots, p_{n-1}$.

The preceding analysis assumes that the interest rate applied during settlement is constant over the entire length of the agreement. This assumption often is not valid. The interest rate varies between settlement periods, in addition to the variability in the index price. When these two variables (interest rate and index price) are positively correlated, the future contract will out-perform a forward contract, hence its price will be higher. If these two variables are negatively correlated the future contract will under-perform relative to a forward contract and thus its price will be lower.

Exercise D5-1 A stock index pays dividends continuously at a rate of 4% annually. The current price of one unit of the index is \$105. What is the price of a prepaid forward contract for one unit of the index 16-months from today?

Exercise D5-2 A stock has current price of \$80. A dividend of \$4 is expected 9-months from today. The risk-free rate of interest is 6% annual effective. Find the forward price of a one-year forward contract on one share of stock.

Exercise D5-3 The current price of a stock is \$90. It is expected to pay dividends continuously at a constant annual rate of 3%. The risk-free force of interest is 5%. Find the forward price of a 15-month forward contract for one share.

Exercise D5-4 You buy a 6-month futures contract on 1000 units of an index, which has a current price of \$800 per unit. Your margin fund initial deposit is \$200,000. This fund is credited with interest at 4% annual effective. The account is settled monthly. If the price per unit is \$795 one month from today, what is the balance of your margin fund account at that time?