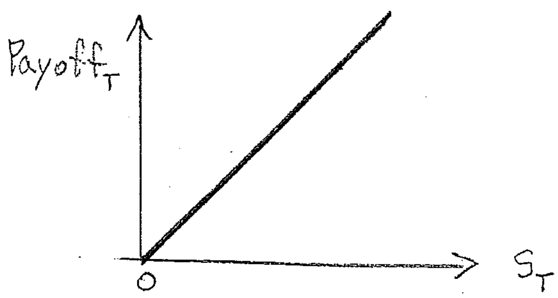
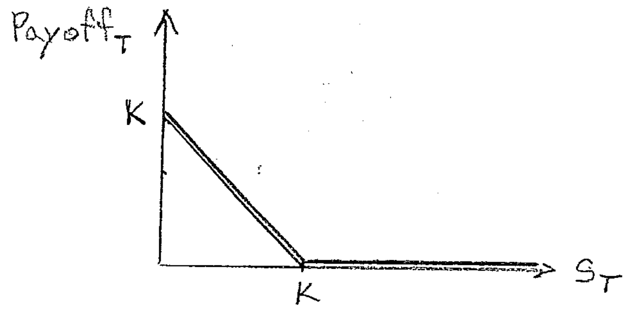
Chapter D3 - Agreement Combinations and Strategies

Position Combinations:

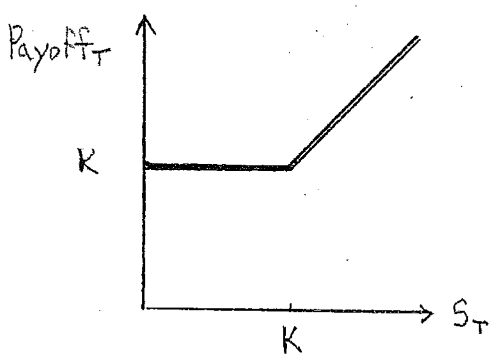
(A) Own an asset worth S_0 (Long position) and want to insure that you get at least K for it at time T. So you buy a Put with strike price K.



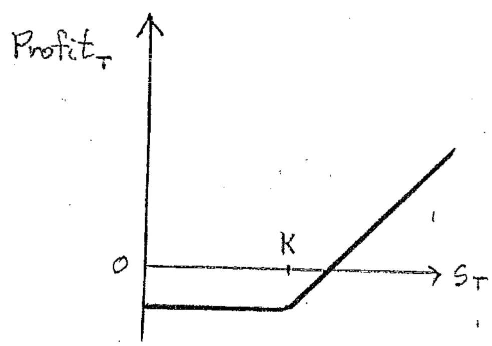
Own Asset



Purchased Put



Combined Positions



Profit of Combination

The Put, costing $P(K,T)$ is viewed as insurance on the value of the asset.

$$Profit_T = Payoff_T - (1+i)^T S_0 - (1+i)^T P(K,T)$$

Underlying Mathematics of Combined Payoff

Adding equations that are each formed by continuous line segments.

(1.) A Sum of continuous functions is continuous

(2.) Adding Line Segments

$$\begin{array}{ll} y_1 = a_1 & \text{horizontal} \\ + \; y_2 = a_2 & \text{horizontal} \\ \hline y = y_1 + y_2 = a_1 + a_2 & \text{horizontal} \end{array}$$

$$\begin{array}{ll} y_1 = a_1 & \text{horizontal} \\ + \; y_2 = mx + a_2 & \text{slope } m \\ \hline y = y_1 + y_2 = mx + (a_1 + a_2) & \\ & \text{slope } m \end{array}$$

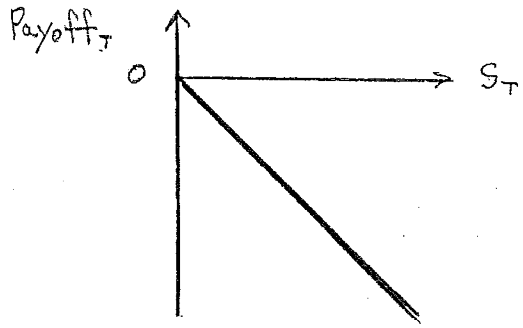
$$\begin{array}{ll} y_1 = x + a_1 & \text{slope } +1 \\ + \; y_2 = -x + a_2 & \text{slope } -1 \\ \hline y = y_1 + y_2 = a_1 + a_2 & \text{horizontal} \end{array}$$

$$\begin{array}{ll} y_1 = m_1 x + a_1 & \text{slope } m_1 \\ + \; y_2 = m_2 x + a_2 & \text{slope } m_2 \\ \vdots & \\ + \; y_n = m_n x + a_n & \text{slope } m_n \\ \hline \end{array}$$

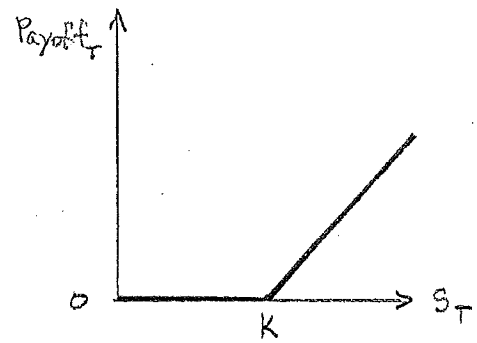
$y =$

(B)

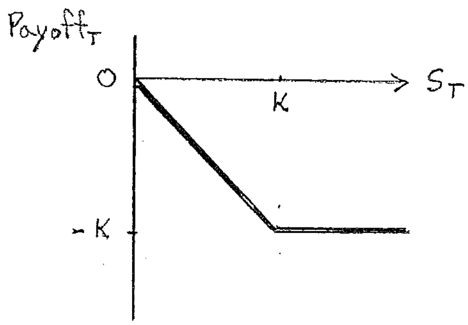
Buy a K-strike Call.

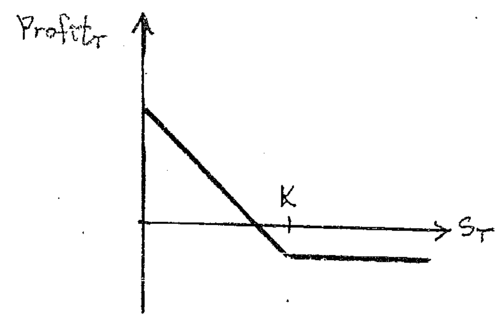


Owe Asset



Buy Call



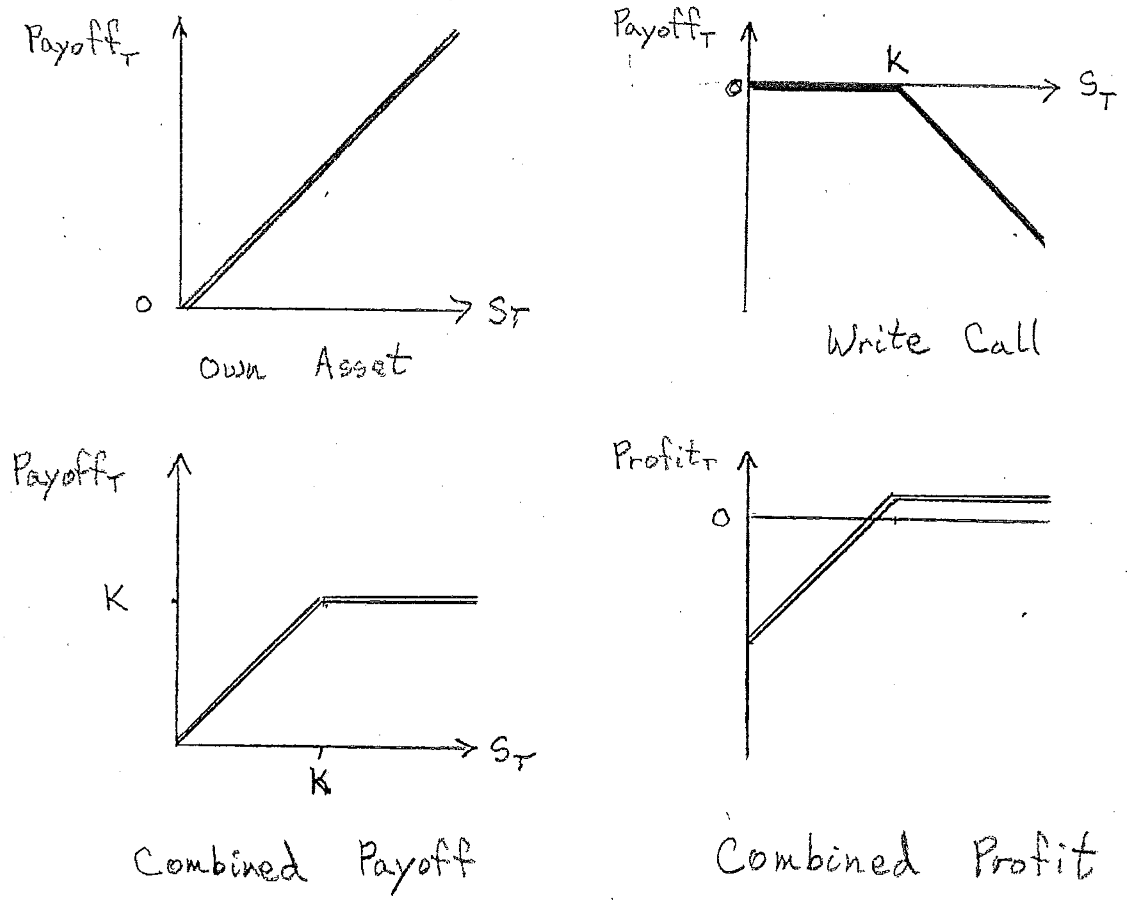


The purchased call places a <u>cap</u> on the exposure that we have in the short position.

$$\text{Profit}_T = \text{Payoff}_T + (1+i)^T S_0 - (1+i)^T C(K,T)$$

where $C(K,T)$ is the cost of the call.

(c) Writing a Covered Call

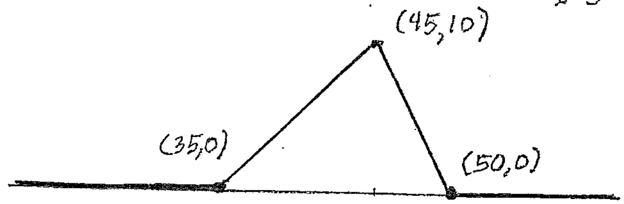


Here

$$\text{Profit}_T = \text{Payoff} - \nu^{-T} S_0 + \nu^{-T} C(K,T)$$

where $C(K,T)$ is the amount received for the Call.

Exercise D3-A

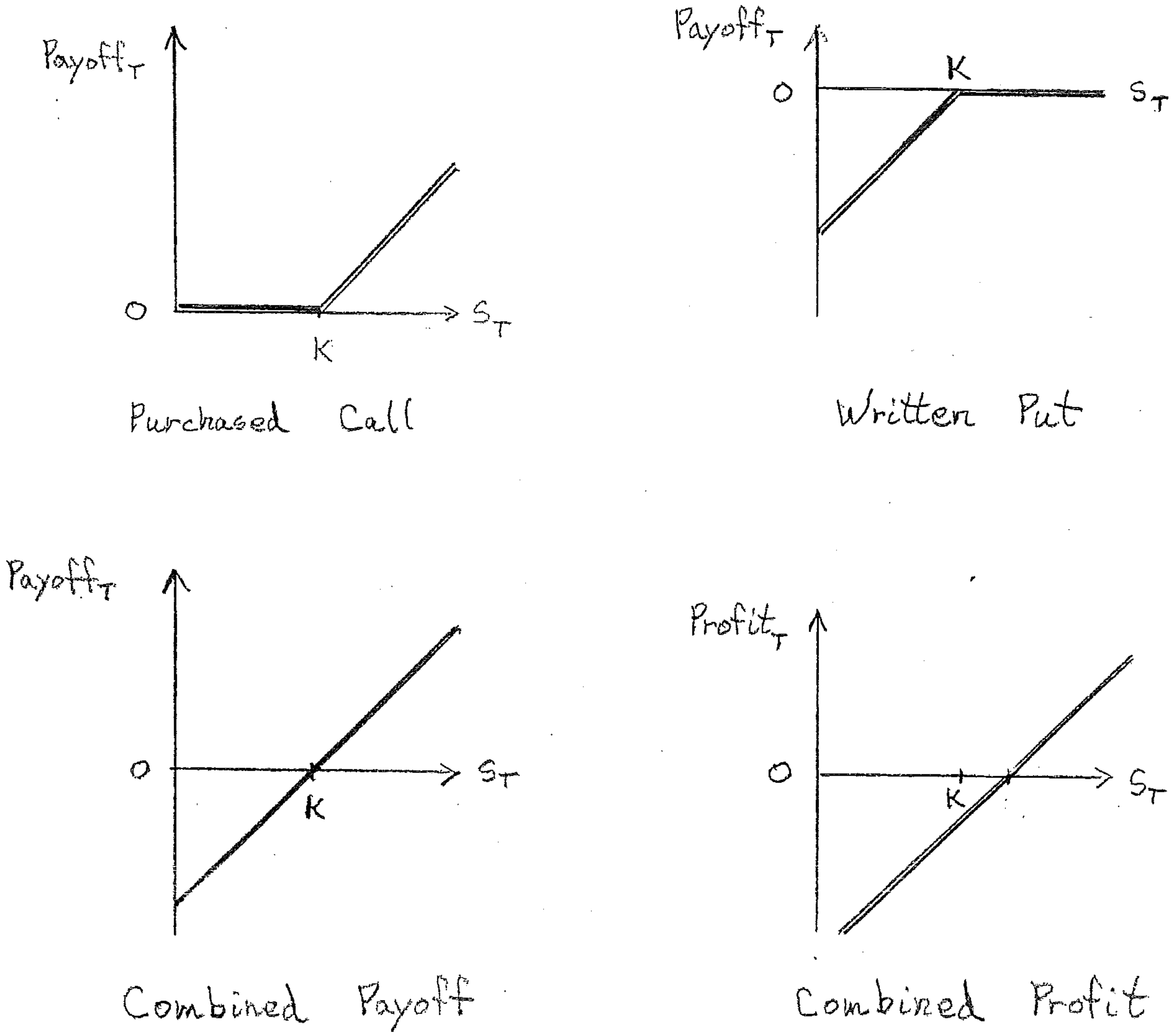


How can we create the payoff diagram on the right by buying or selling (possibly multiple) only call options at various strike prices.

Exercise D3 - B You buy a call with a strike price of \$95 for a premium of \$9.20. You sell a call with a strike price of \$106 and receive a premium of \$4.02. Both agreements begin on the same date and end 9-months later. The interest rate is nomial 4% convertible quarterly. What is the largest profit you can make on this combination?

Put - Call Parity

Purchase a K-strike Call and write a K-strike Put for the same asset with the same expiration date



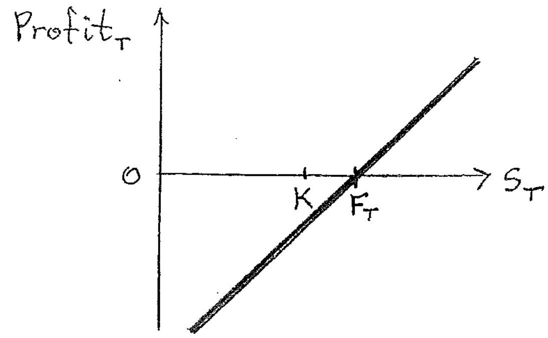
Here

where

$C(K,T) =$ amount paid for call

and $P(K,T) =$ amount received for put.

Note that this is also the Profit diagram of a (no premium) forward contract on the asset with forward price $F_{0,T}$



That is, prices will always gravitate to levels that give equivalent positions equivalent costs.

For the no premium forward contract with forward price $F_{0,T}$ on the same asset with the same expiration date, the profit line is:

$$Profit_T = Payoff_T - 0$$

$$= S_T - F_{0,T}$$

For the (purchase K-strike Call, write K-strike Put) combination position the profit function is

$$\text{Profit}_T = S_T - \left(K + \nu^{-T}\{C(K,T) - P(K,T)\}\right)$$

Under No Arbitrage Pricing these two should be equivalent. That is,

$$F_{0,T} = K + \nu^{-T}\{C(K,T) - P(K,T)\}$$

or

Put - Call Parity Equation

This equation relates the prices of a K-strike Put, a K-strike Call and a no premium $F_{0,T}$-strike forward contract to one another. The right side of this equation is the present value of $F_{0,T} - K$ assessed via the risk-free rate of interest.

To enter into a forward contract you have to find someone willing to sell the asset at time T. If the asset has value S_0 now, you can find someone willing to sell it, if you offer

$$S_0(1+i)^T = \nu^{-T} S_0$$

as the forward strike price $F_{0,T}$.

Putting this value for $F_{0,T}$ into the put-call parity equation produces

which is often a more useful form of the Put-Call parity relationship.

Exercise D3-C

A three-month call option with an exercise price of \$50 sells for \$1. The current stock price is \$49. The interest rate is 9% convertible monthly. Find the price of a three-month put option with the same exercise price of \$50.

Exercise D3 - D Given the following about one-year derivatives on a particular asset:

Forward price is \$163.13

\$150-strike call premium is \$23.86

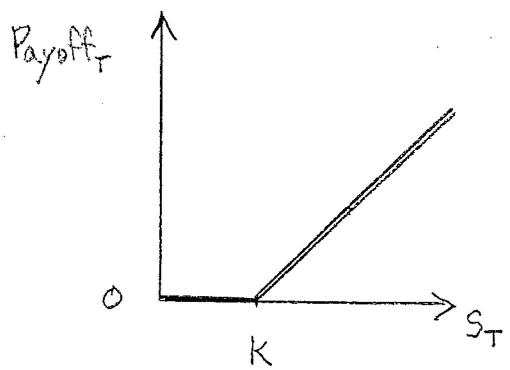
\$150-strike put premium is \$11.79

Find the risk-free annual effective rate of interest.

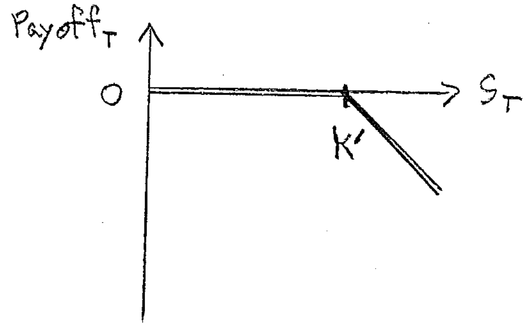
Strategic Combinations of Options

Bull Spreads –

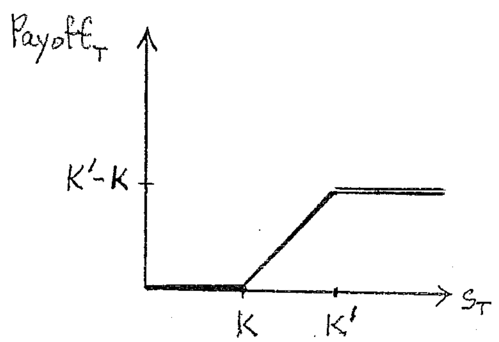
(A) Buy a K-strike Call, Write a K′-strike Call, with $K < K'$.



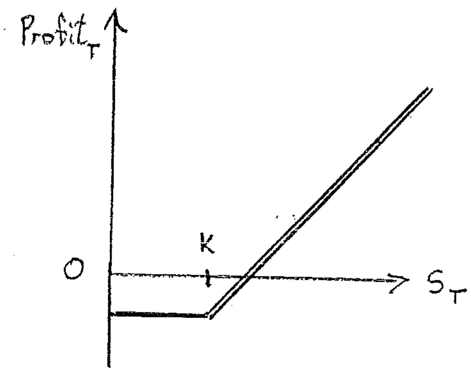
Buy K-Strike Call



Write a K′-Strike Call

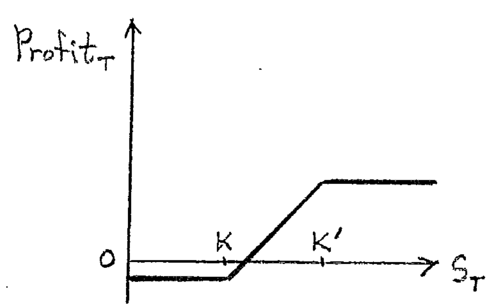


Combined Payoff



Profit of Buy K-Strike Call

Combined position risks (gains) less if asset value falls below K (above K′), in comparison to Buy K-strike call.



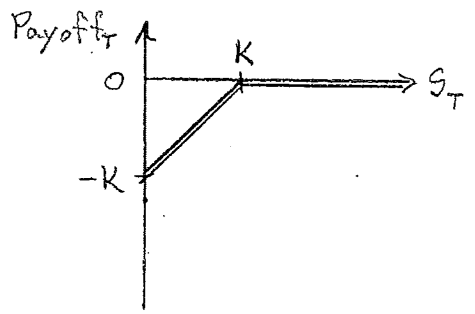
Profit of Combined Position

(B) (You check) A similar profit function results if you Buy a K-strike Put and Sell K′-strike Put $(K < K')$.

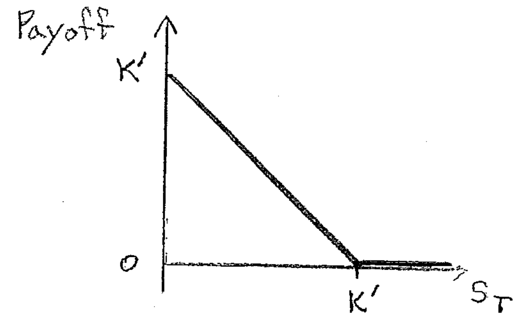
Bear Spreads –

(A′) Write a K-strike Call, Buy a K′-strike Call (check

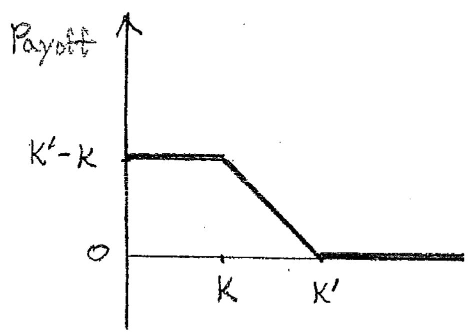
(B′) Write a K-strike Put, Buy a K′-strike Put $(K < K')$



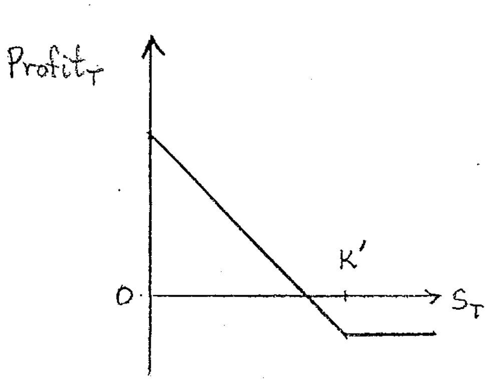
Write K-strike Put



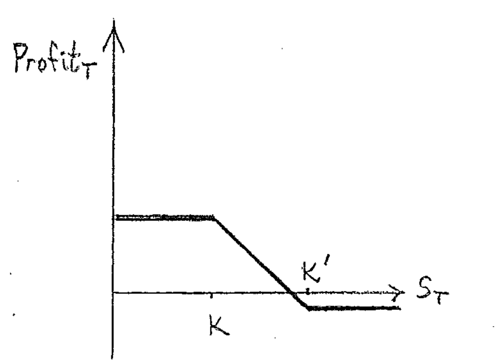
Buy K′-strike Put



Combined Payoff



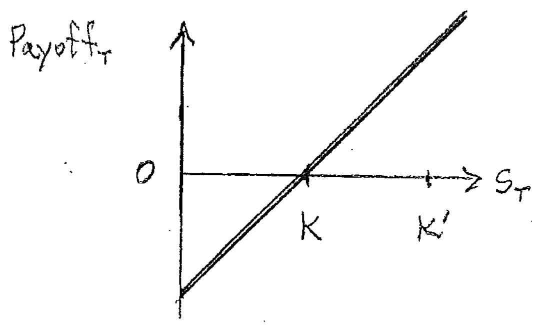
Profit of Buy K′-strike Put



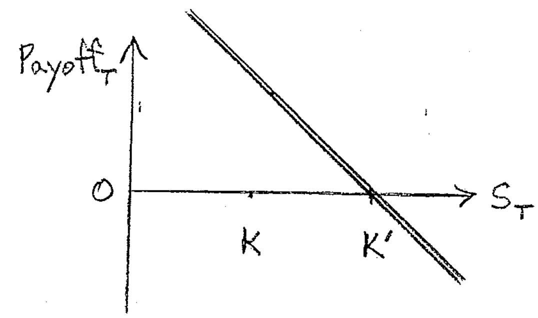
Profit of Combined Position

Box Spread $(K < K')$

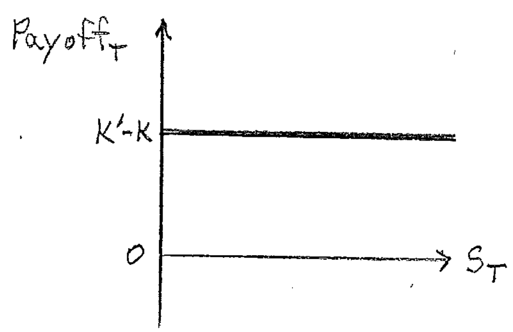
Both:
(a) Buy a K-strike call and Sell a K-strike put
(b) Buy a K'-strike put and Sell a K'-strike call



Combined Payoff of (a)



Combined Payoff of (b)

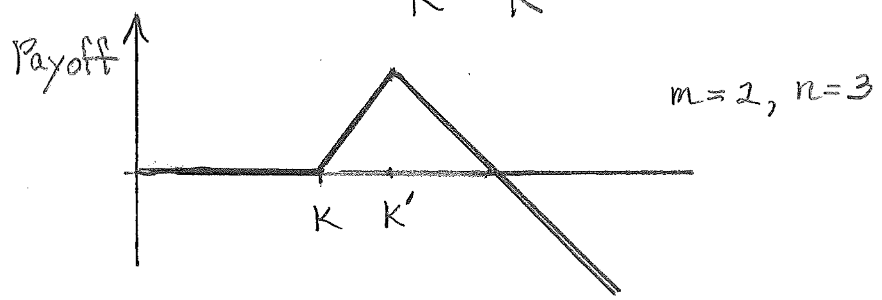


The prices of the puts and calls factor into the profit, but the true benefit may be in taxation advantages.

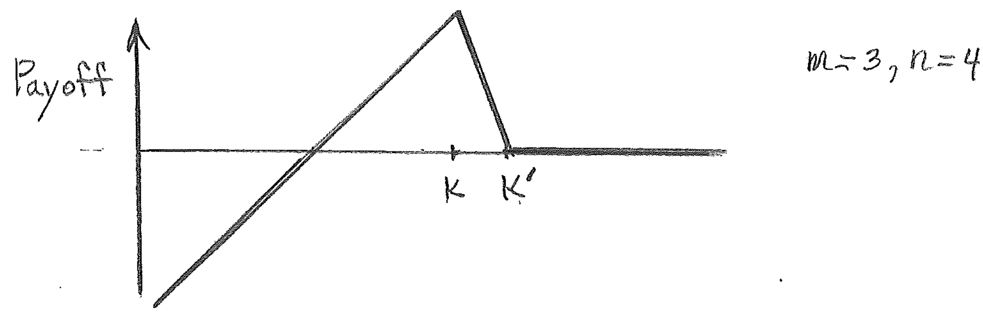
Ratio Spreads

(A) Buy m K-strike Calls, Write n K′-strike Calls.

$K < K'$



(B)

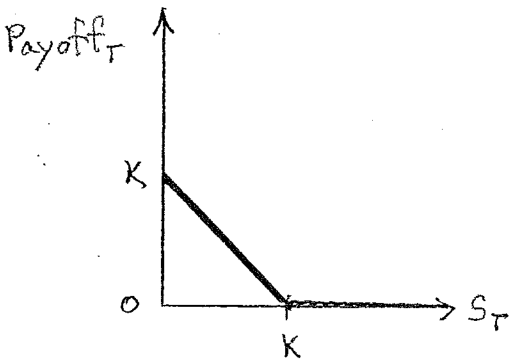


A no cost ratio spread might be constructed by choosing m, n, K and K' so that the net premium is zero, i.e.

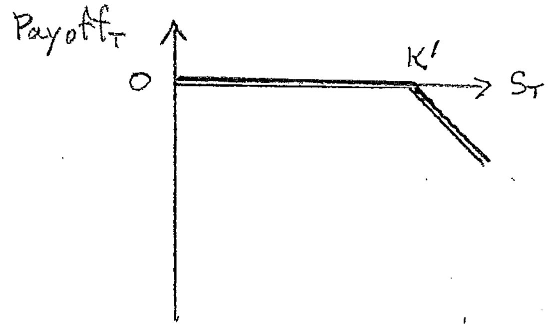
$$m(\text{Buy prem. paid}) = n(\text{Sell prem. received}).$$

In this case profit ≡ payoff.

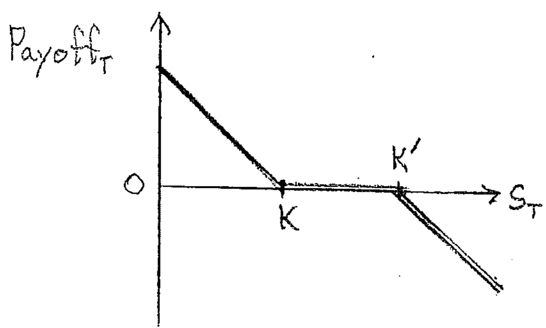
Collar –



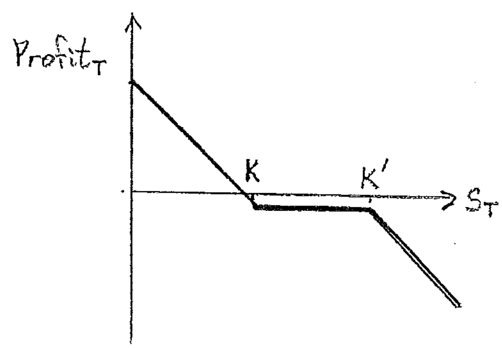
Purchased K-strike Put



Written K'-strike Call



Combined Position



Profit of Combined Position

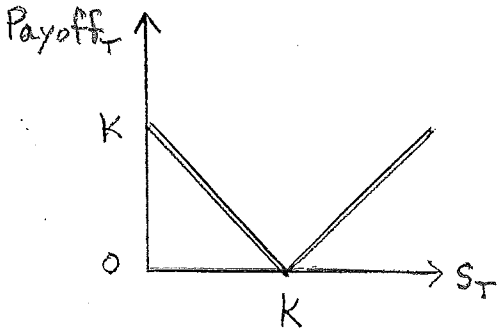
Here

$$\text{Profit}_T = \text{Payoff}_T - \nu^{-T}\left\{P(K,T) - C(K',T)\right\}$$

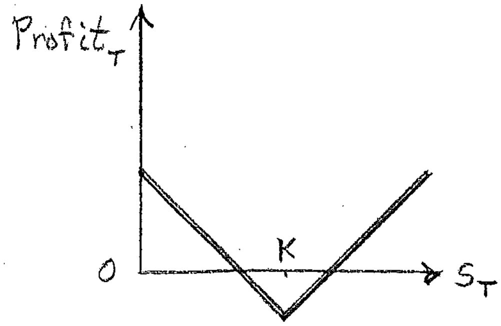
and the collar width is $K'-K$. Sometimes in the choice of K (and K'), it is possible to make $P(K,T) = C(K',T)$. This is a "no-cost collar".

Strategies Based on Volatility

Stradle –



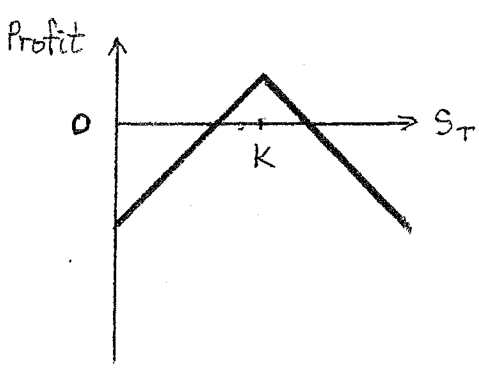
Combined Payoff



Combined Profit

Written Stradle – Speculating that the value will be in the vicinity of K at time T.

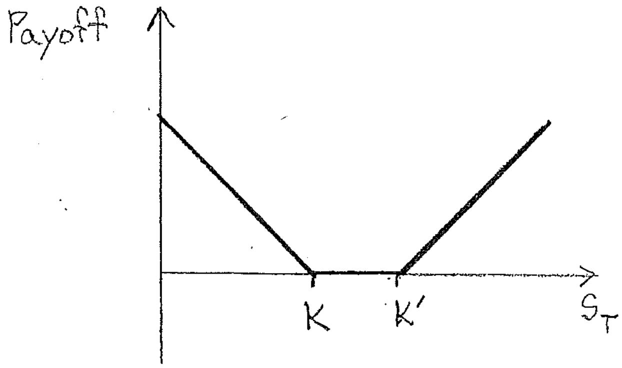
Sell a K-strike Call and Sell a K-strike Put

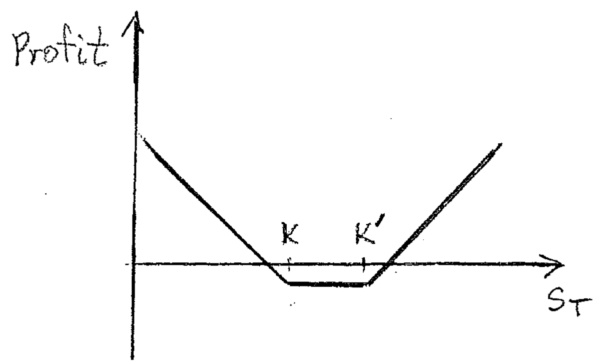


Strangle

$K < K'$

Buy a K-Strike Put and Buy a K'-Strike Call





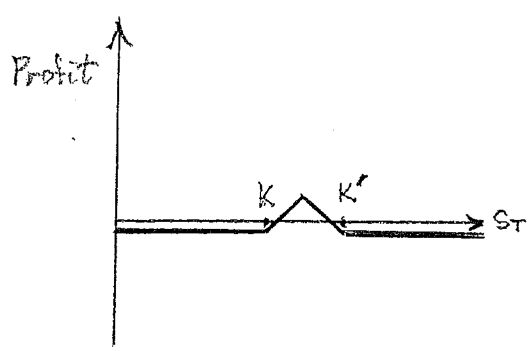
Butterfly Spread

Buy a $K < K'$ Strangle and

Sell (Write) a $\frac{K+K'}{2}$ Stradle

Strangle Profit (above right)
Stradle Profit (bottom previous page at $\frac{K+K'}{2}$, not K)

Combined Profit:



Exercise D3-E

Buy a \$150-strike one year call for \$2.75
Buy a \$110-strike one year put for \$1.90.
The risk-free rate of interest is 4% annual effective.

(a) Draw the payoff and profit diagrams for this setting.

(b) What is this shape called and when is it advantageous?

(c) What is the profit if the spot price is \$100 at expiration.

Lognormal Distribution

If X has $N(\mu, \sigma^2)$ distribution

then $Y = e^X$ has a lognormal distribution

Density of Y is:

$$f(y) = \begin{cases} \dfrac{1}{y\sqrt{2\pi}\,\sigma}\, e^{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}} & \text{if } y>0 \\ 0 & \text{if } y \le 0 \end{cases}$$

Here the parameters satisfy:

$-\infty < \mu < \infty$ and $\sigma > 0$

The distribution function of Y is:

$$F(y) = P[Y \le y] = \begin{cases} \Phi\left(\dfrac{\ln(y)-\mu}{\sigma}\right) & y>0 \\ 0 & \text{if } y \le 0 \end{cases}$$

where $\Phi(t)$ is the dist. function of $N(0,1)$.

$$E[Y] = e^{\mu + \frac{\sigma^2}{2}} \qquad Var[Y] = e^{(2\mu+\sigma^2)}\left[e^{\sigma^2} - 1\right]$$

$$\text{Median}(Y) = e^{\mu} \qquad \text{Mode}(Y) = e^{\mu - \sigma^2}$$

Assumptions behind Black – Scholes formula:

Let $S \equiv S_0$ denote the current stock price (Known)

and S_T denote the stock price at expiry (unknown).

① Model S_T by assuming:

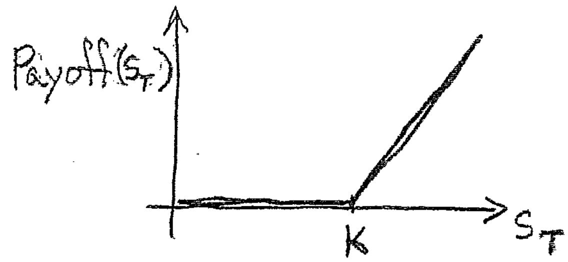
$\ln\left(\frac{S_T}{S_0}\right)$ has a normal distribution with mean μT and variance $\sigma^2 T$, i.e.

$\frac{S_T}{S_0}$ is lognormal.

② Assume the current stock price is the expected value of the present value at expiry. That is

$$\boxed{S_0 = E\left[e^{-\delta T} S_T\right]}$$

The payoff of a K-Strike Call is:



The expected value of the present value of the unknown Payoff(S_T) is:

$$\boxed{C_0 = \underset{Ave}{E}\left[e^{-\delta T} \text{Payoff}(S_T) \right] \quad \text{Today's Price of Call}}$$

C_0 = Expected present value of the future payoff.

S_0 = Current Price

K = Exercise (Strike) Price

δ = risk-free force of interest ($\ln(1+i)$), with i, the interest on U.S. Tresury bonds.

$$C_0 = e^{-\delta T} \underset{Ave.}{E}\left[\text{Payoff}(S_t)\right]$$

$$= e^{-\delta T} \int_K^\infty (x - K) g_T(x)\, dx$$

where $g_T(x)$ denotes the density of S_T.

a math-stat exercise* then shows that

$$\boxed{C_0 = S_0\,\Phi(d_1) - K e^{-\delta T}\,\Phi(d_2)}$$

Black-Scholes Formula

where

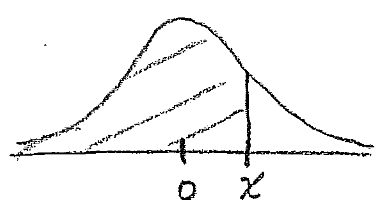
$$d_1 = \frac{\ln(S_0/K) + \left(\delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

and

$$d_2 = \frac{\ln(S_0/K) + \left(\delta - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

where

$\Phi(x)$ is distribution function of a $N(0,1)$



and

σ is the standard deviation of $\ln(S_T/S_0)$.

Lemma A If $Y \sim N(\mu, \sigma^2)$ (density $g_Y(y)$)

$$\int_a^\infty e^y g_Y(y)\,dy = e^{\mu + \frac{\sigma^2}{2}} \Phi\left(\frac{\mu + \sigma^2 - a}{\sigma}\right)$$

$$\int_a^\infty e^y g_Y(y)\,dy = \int_a^\infty e^y \phi\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma}\,dy$$

$\phi(t)$ density of $N(0,1)$

$$= \int_{\frac{a-\mu}{\sigma}}^\infty e^{\mu + \sigma z}\phi(z)\,dz$$

$z = \frac{y-\mu}{\sigma}$

$\mu + \sigma z = y$

$\sigma dz = dy$

$$= e^{\mu} \int_{\frac{a-\mu}{\sigma}}^\infty e^{\sigma z} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,dz$$

$$= e^{\mu} \int_{\frac{a-\mu}{\sigma}}^\infty \frac{e^{-\frac{1}{2}(z^2 - 2\sigma z + \sigma^2)}}{\sqrt{2\pi}} e^{\frac{1}{2}\sigma^2}\,dz$$

$$= e^{\mu + \frac{\sigma^2}{2}} \int_{\left(\frac{a-\mu-\sigma^2}{\sigma}\right)}^\infty \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\,ds$$

$s = z - \sigma$

$ds = dz$

$$= e^{\mu + \frac{\sigma^2}{2}} \Phi\left(\frac{\mu + \sigma^2 - a}{\sigma}\right)$$

where $\Phi(t)$ is the distribution function of a $N(0,1)$ R.V.

Lemma B If $g_T(x)$ denotes the density of S_T, then

(a) $$g_r(t) = g_T(ts_0)\, s_0$$

where $g_r(\cdot)$ is the density of S_T/S_0

and

(b) $$g_L(w) = e^w \cdot g_r(e^w)$$

is the density of $\ln(S_T/S_0)$.

Dist. Functions:

$$G_r(t) = P\left[\left(S_T/S_0\right) \le t\right] = P\left[S_T \le S_0 t\right]$$

$$= G_T(S_0 t) \quad \text{so}$$

so densities satisfy

$$g_r(t) = g_T(tS_0)\, S_0$$

Dist. Functions:

$$G_L(w) = P\left[\ln(S_T/S_0) \le w\right] = P\left[\left(S_T/S_0\right) \le e^w\right]$$

$$= G_r(e^w)$$

so densities satisfy

$$g_L(w) = g_r(e^w)\, e^w$$

Black-Scholes Formula

From assumption 2:

$$S_0 = E\left[e^{-\delta T} S_T\right] = e^{-\delta T} E\left[S_T\right]$$

$$= e^{-\delta T} S_0 \, E\left[S_T/S_0\right]$$

$$= S_0 e^{-\delta T} e^{\mu T + \frac{\sigma^2 T}{2}}$$ pg D3-14 & 15

or

$$1 = e^{T(\mu + \frac{\sigma^2}{2} - \delta)}$$

or

$$\mu + \frac{\sigma^2}{2} - \delta = 0$$

or

$$\mu = \delta - \frac{\sigma^2}{2} \qquad (1^*)$$

From page D3-16:

$$C_0 = e^{-\delta T} \int_K^\infty (x - K) g_T(x) \, dx$$

$$= e^{-\delta T} \int_{(K/S_0)}^\infty (S_0 t - K) g_T(S_0 t) S_0 \, dt$$

$t = x/S_0$
$S_0 t = x$
$S_0 dt = dx$

$$= e^{-\delta T} \int_{(K/S_0)}^\infty (S_0 t - K) g_r(t) \, dt$$ Lemma B (a)

or

$$C_0 = e^{-\delta T} \int_{\ln(K/S_0)}^{\infty} (S_0 e^w - K)\, g_T(e^w)\, e^w \, dw \qquad \begin{array}{l} t = e^w \\ dt = e^w dw \end{array}$$

$$= e^{-\delta T} \int_{\ln(K/S_0)}^{\infty} (S_0 e^w - K)\, g_L(w)\, dw \qquad \text{Lemma B (b)}$$

But $L = \ln(S_T/S_0) \sim N(\mu T, \sigma^2 T)$. Therefore,

$$C_0 = S_0 e^{-\delta T} \int_{\ln(K/S_0)}^{\infty} e^w g_L(w)\, dw - K e^{-\delta T} \int_{\ln(K/S_0)}^{\infty} g_L(w)\, dw$$

$$= S_0 e^{-\delta T} e^{\mu T + \frac{\sigma^2 T}{2}} \Phi\left(\frac{\mu T + \sigma^2 T - \ln(K/S_0)}{\sqrt{\sigma^2 T}}\right) \qquad \text{by Lemma A}$$

$$- K e^{-\delta T} \left\{ \Phi\left(\frac{\mu T - \ln(K/S_0)}{\sqrt{\sigma^2 T}}\right) \right\}$$

From 1*, $\mu = \delta - \frac{\sigma^2}{2}$, so

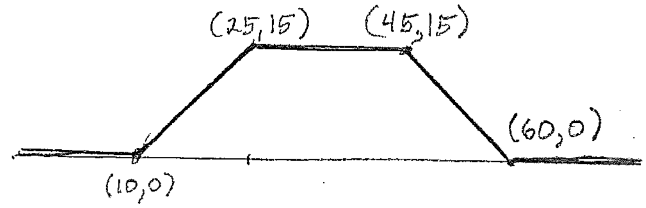
$$C_0 = S_0 \Phi\left(\frac{\ln(S_0/K) + (\delta + \frac{\sigma^2}{2})T}{\sqrt{\sigma^2 T}}\right)$$

$$- K e^{-\delta T} \Phi\left(\frac{\ln(S_0/K) + (\delta - \frac{\sigma^2}{2})T}{\sqrt{\sigma^2 T}}\right)$$

which is the Black-Scholes Formula.

Exercise D3-1

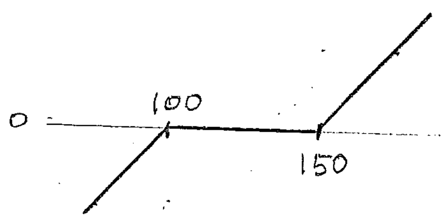
What combination of purchased or written calls creates a payoff diagram as shown on the right.



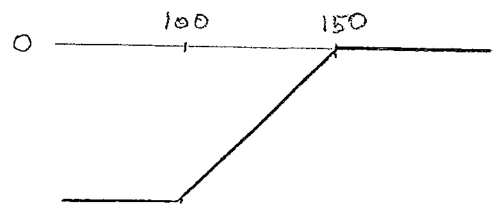
Exercise D3-2

What combo of calls and/or puts produce each payoff shape shown below:

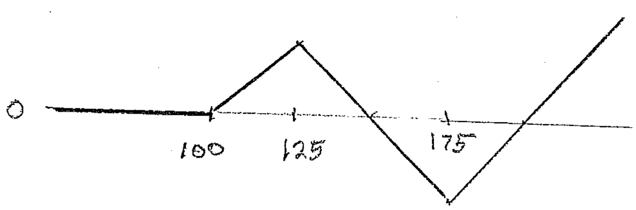
(a)



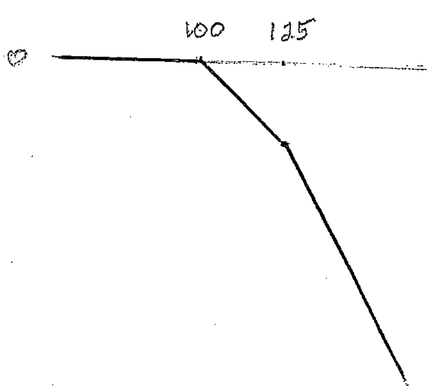
(b)



(c)



(d)



(e) Use diagram from Exercise D3-1 and only use puts.

Exercise D3 - 3 For a certain asset we are given

(i) a 4 year \$585-strike call has a premium of \$56

(ii) a 4 year \$585-strike put has a premium of \$32

(iii) also the risk-free interest rate is 4.2% annual effective.

What is the current value of this asset?

Exercise D3-4 You believe that the volatility of a certain commodity is much lower than the market assesses it to be. Which of the following is your best investment strategy:

(A) Buy a bull-spread (B) Write a bull-spread

(C) Buy a straddle (D) Write a straddle

(E) Buy a strangle

Exercise D3-5 Which of the following combination positions has an unlimited potential loss?

(A) Buy a bull spread (B) Buy a bear spread

(C) Buy a collar (D) Buy a box spread