

Chapter 11 - Duration, Convexity and Immunization

Section 11.2 - Duration

Consider two opportunities for an investment of \$1,000.

A: Pays \$610 at the end of year 1 and \$1,000 at the end of year 3

B: Pays \$450 at the end of year 1, \$600 at the end of year 2 and \$500 at the end of year 3.

Both have a yield rate of $i = .25$ because $(1.25)^{-1} = .8$,

$$1000 = (.8)(610) + (.8)^3(1000)$$

and

$$1000 = (.8)(450) + (.8)^2(600) + (.8)^3(500).$$

The repayment patterns of these two investments are quite different and we seek to compare them on the basis of the **timing of the repayments**. Getting the repayments sooner would be advantageous if reinvestment yield rates are above the current yield of this investment (in the above setting $i = .25$), whereas delaying the repayments is advantageous if the reinvestment interest rates are lower than the current yield rate.

Method of Equated Time (See section 2.4) provides a simple answer to measure the timing of the repayments:

Here R_t denotes a return ($R_t > 0$ is a payment back to the investor made at time t).

Example: (from page 11-1)

$$A: \quad \bar{t} = \frac{1(610)+3(1000)}{610+1000} = 2.24$$

$$B: \quad \bar{t} = \frac{1(450)+2(600)+3(500)}{450+600+500} = 2.03.$$

The money is returned faster under investment B.

A better index would also take into account the **current value** of the future repayments:

Macaulay Duration:

Here the investment yield rate is used in ν . The quantity \bar{d} is a decreasing function of i .

Example: (from page 11-1)

$$A: \quad \bar{d} = \frac{1(.8)(610)+3(.8)^3(1000)}{(.8)(610)+(.8)^3(1000)} = 2.024$$

$$B: \quad \bar{d} = \frac{1(.8)(450)+2(.8)^2(600)+3(.8)^3(500)}{(.8)(450)+(.8)^2(600)+(.8)^3(500)} = 1.896.$$

Both \bar{t} and \bar{d} are **weighted averages of the return times**. In \bar{t} the weights are the **return amounts** and in \bar{d} the weights are the

The **(net) present value of a set of returns** is:

It represents the value of an investment today. We now focus on it as a **function of the current interest rate i** .

Since interest rates frequently change, the volatility of the present value to changes in i is very important. It is measured with

Volatility:

The minus sign is included because $P(i)$ is a decreasing function of i and hence $P'(i) < 0$. So including the minus sign makes the \bar{v} value positive and therefore makes larger values of \bar{v} indicate more volatility (susceptibility to changes in i), relative to the magnitude of $P(i)$.

We now relate volatility to duration by examining their defining expressions.

$$\bar{v} \equiv -\frac{P'(i)}{P(i)}$$

$$= \frac{-\frac{d}{di}(\sum_{t=1}^n \frac{1}{(1+i)^t} R_t)}{(\sum_{t=1}^n \frac{1}{(1+i)^t} R_t)}$$

Thus our measure of **volatility** \bar{v} is often called **modified duration**, even though its purpose is quite different from that of duration.

Example: (from page 11-1)

$$A: \quad \bar{v} = (.8)(2.024) = 1.6192$$

$$B: \quad \bar{v} = (.8)(1.896) = 1.517.$$

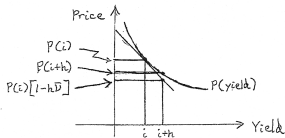
It does make sense that an investment that takes longer to achieve its return will be more susceptible to changes in the interest rate i . Note also that by the definition of \bar{v} ,

$$P'(i) = -P(i)\bar{v} \quad \text{implies}$$

$$\lim_{h \rightarrow 0} \frac{P(i+h) - P(i)}{h} = -P(i)\bar{v} \quad \text{or}$$

$$P(i+h) - P(i) \doteq -h\bar{v}P(i) \quad \text{which produces}$$

when h is small. Typically, this approximation produces a value that is below the actual value of $P(i+h)$ when $h \neq 0$.



With continuous compounding at a constant force of interest δ ,

$$\bar{d} = \bar{\nu} = \frac{\sum_{t=1}^n te^{-\delta t} R_t}{(\sum_{t=1}^n e^{-\delta t} R_t)}$$

that is, Macaulay duration and modified duration are the same. See pages 455-456 in the textbook.

Example:

Consider a **zero coupon bond** that makes one payment of C at the end of n periods, with a effective interest rate of i for each period.

$$\bar{t} = \frac{nC}{C} = n$$

$$\bar{d} = \frac{n\nu^n C}{\nu^n C} = n$$

$$\bar{\nu} = n\nu$$

Example:

Consider an **annuity immediate** with payments of k at the end of each of n periods and an interest rate of i .

$$\bar{t} = \frac{\sum_{t=1}^n tk}{\sum_{t=1}^n k} = \frac{k \frac{n(n+1)}{2}}{nk} = \frac{n+1}{2}$$

$$\bar{d} = \frac{\sum_{t=1}^n t\nu^t k}{\sum_{t=1}^n \nu^t k}$$

$$\bar{v} = \bar{d}\nu$$

Example:

Consider a **perpetuity immediate** with payments of k at the end of each period and an interest rate of i and $\nu = \frac{1}{1+i}$.

$$\bar{t} = \frac{\sum_{t=1}^{\infty} tk}{\sum_{t=1}^{\infty} k} \quad (\text{This is undefined.})$$

$$\bar{d} = \frac{\sum_{t=1}^{\infty} t\nu^t k}{\sum_{t=1}^{\infty} \nu^t k} = \frac{\nu \sum_{t=1}^{\infty} t\nu^{t-1}}{\lim_{n \rightarrow \infty} \left(\frac{\nu(1-\nu^n)}{(1-\nu)} \right)} = (1-\nu) \sum_{t=1}^{\infty} \left[\frac{d}{d\nu} \nu^t \right]$$

$$= (1-\nu) \frac{d}{d\nu} \left[\sum_{t=1}^{\infty} \nu^t \right] = (1-\nu) \frac{d}{d\nu} \left[\frac{\nu}{1-\nu} \right] = (1-\nu) \frac{(1-\nu) + \nu}{(1-\nu)^2}$$

Exercise 11-6: The current price of an annual coupon bond is 100. The derivative of the price of the bond with respect to the yield to maturity is -650. The yield to maturity is an effective rate of 7%.

- (a) Calculate the Macaulay duration of the bond.
- (b) Estimate the price of the bond using the approximation formula on page 11-7 when the yield is 8% instead of 7%.

Section 11.2 - Convexity

Typically the present value of a set of cash flows decreases as a function of the interest rate i . In fact, this function is most often a **convex decreasing function**. A second order Taylor series expansion will capture the curvature in addition to the trend and will often well approximate the changes in the function as i changes, at least for small changes in i . In the previous section we let

$$\bar{v} = -\frac{P'(i)}{P(i)}$$

where the minus sign was inserted because $P'(i)$ is usually negative. Similarly, we now let

which is called the **convexity of the present value of the cash flow**.

The second order Taylor series approximation of $P(i)$ then produces

We also note that

$$\begin{aligned}\frac{d\bar{v}}{di} &= \frac{d}{di} \left[-\frac{P'(i)}{P(i)} \right] \\ &= \frac{-P(i)P''(i) + [P'(i)]^2}{[P(i)]^2}\end{aligned}$$

Recall that \bar{v} describes the sensitivity of $P(i)$ to changes in i . Likewise, \bar{c} plays a role in describing the sensitivity of \bar{v} to changes in i .

Note that

$$P(i) = \sum_{t=1}^n (1+i)^{-t} R_t,$$

$$P'(i) = \sum_{t=1}^n -t(1+i)^{-(t+1)} R_t \quad \text{and}$$

$$P''(i) = \sum_{t=1}^n t(t+1)(1+i)^{-(t+2)} R_t.$$

Example: Annuity Immediate (See page 11-9)

$$\bar{c} = \frac{\sum_{t=1}^n t(t+1)\nu^{t+2}k}{\sum_{t=1}^n \nu^t k} = \frac{(1-\nu)}{\nu(1-\nu^n)} \sum_{t=1}^n t(t+1)\nu^{t+2}$$

In continuous compounding settings with a constant force of interest described by δ ,

$$P(\delta) = \sum_{t=1}^n e^{-\delta t} R_t,$$

$$P'(\delta) = \sum_{t=1}^n t e^{-\delta t} R_t \quad \text{and}$$

$$P''(\delta) = \sum_{t=1}^n t^2 e^{-\delta t} R_t.$$

In these settings, the **Macaulay convexity** is defined as:

$$\frac{P''(\delta)}{P(\delta)} = \frac{\sum_{t=1}^n t^2 e^{-\delta t} R_t}{\sum_{t=1}^n e^{-\delta t} R_t}$$

Exercise 11-11: A loan is to be repaid with payments of \$1,000 at the end of year 1, \$2,000 at the end of year 2, and \$3,000 at the end of year 3. The effective rate of interest is $i = .25$. Find (a) the amount of the loan, (b) the duration, (c) the modified duration, and (d) the convexity.

Section 11.4 - Interest Sensitive Cash Flows

Some cash flow settings have present values that are quite sensitive to changes in i because the returns themselves depend on i .

Examples are callable bonds and mortgages without a prepayment penalty. To better capture the volatile nature of the present value, the function $P'(i)$ is approximated via

$$P'(i) \doteq \frac{P(i+h) - P(i-h)}{2h}$$

and for small h the **effective volatility** is described by

where the order in the numerator is reversed to make the ratio positive.

Similarly for small h , the **effective convexity** is described by

$$\bar{c}_e \doteq \frac{\frac{P(i-h)-P(i)}{h} - \frac{P(i)-P(i-h)}{h}}{hP(i)}$$

Again the order of these differences is chosen to make this positive, since $P(i-h) + P(i+h) > 2P(i)$ for a decreasing convex function.

Example:

A homebuyer takes out a 30-year \$100,000 loan at 6% convertible monthly. At the end of 15 years, the homebuyer can pay off the loan if interest rates fall, but will keep the existing loan if they rise or stay the same. Find \bar{d}_e and \bar{c}_e using 7% and 5%, that is $h = .01$.

$$100,000 = P(.06) = \sum_{t=1}^{360} (1.005)^{-t} (\text{monthly pmt}) \quad \text{produces}$$

$$(\text{monthly pmt}) = \frac{100,000}{\sum_{t=1}^{360} (1.005)^{-t}} = \frac{(1 - \nu)100,000}{\nu(1 - \nu^{360})} = 599.55$$

where here $\nu = (1 + \frac{.06}{12})^{-1} = (1.005)^{-1}$.

We then compute

$$P(.07) = \sum_{t=1}^{360} (1 + \frac{.07}{12})^{-t} (599.55) = 90,116.90.$$

In addition we find

$$= 111,685.14$$

where we have used the outstanding loan balance at the end of 15 years to be

$$(599.55)a_{\overline{180}|.05} = 75,816.24.$$

It follows that

$$\bar{d}_e \doteq \frac{111,685.14 - 90,116.90}{(.02)(100,000)} = 10.784 \quad \text{and}$$

$$\bar{c}_e \doteq \frac{111,685.14 + 90,116.90 - 2(100,000)}{(.01)^2(100,000)} = 180.204.$$

Section 11.5 - Analysis of Portfolios

Companies, investment funds, etc. all have multiple securities, each of which produce a separate cash flow. The present value of the portfolio is the sum of the present values of the securities that comprise it, that is

$$P = P_1(i_1) + P_2(i_2) + \cdots + P_m(i_m),$$

with each security having its individual yield rate. The **modified duration of the portfolio** is :

$$\begin{aligned}\bar{v} &= \frac{-P'}{P} = \frac{P_1(i_1)}{P} \left(\frac{-P'_1(i_1)}{P_1(i_1)} \right) + \cdots + \frac{P_m(i_m)}{P} \left(\frac{-P'_m(i_m)}{P_m(i_m)} \right) \\ &= \frac{P_1(i_1)}{P} (\bar{v}_1) + \cdots + \frac{P_m(i_m)}{P} (\bar{v}_m)\end{aligned}$$

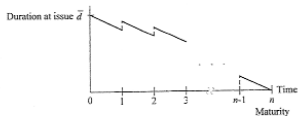
which is a **weighted average of the modified durations** with weights that are the fraction of the total present value in the individual security.

Also the **convexity of the portfolio** becomes

$$\bar{c} = \frac{P''}{P} = \frac{P_1(i_1)}{P} (\bar{c}_1) + \dots + \frac{P_m(i_m)}{P} (\bar{c}_m),$$

a weighted average of the individual convexities.

When assessing a portfolio, separate securities have different start dates and conversion periods. Thus it becomes necessary to measure duration at any point in time, not just at start dates or conversion periods. When measuring duration of any single security, we note that its duration decreases over time. We also note that, as an average time until future payment, the duration increases slightly right after a payment is made, creating a zig-zag plot of \bar{d} over time.



Since the securities in a portfolio differ in their yield rates and conversion periods, it is difficult to measure the effect of increasing i by 100 basis points (100 basis points = 1%). So when assessing a portfolio, it is typical to first standardize the conversion periods and yield rates to annual values before they are altered.

Exercise 11-22: A \$60K portfolio is constructed with \$10K used to buy 2-year zero coupon bonds, \$20 used to buy 5-year zero coupon bonds and \$30K used to buy 10-year zero coupon bonds. The yield rates of the bonds are unknown. Calculate the Macaulay convexity of the portfolio at inception.

$$\begin{aligned}\bar{c} &= \frac{10}{60} \left(\frac{2^2 e^{2\delta_1} 10}{e^{2\delta_1} 10} \right) + \frac{20}{60} \left(\frac{5^2 e^{5\delta_2} 20}{e^{5\delta_2} 20} \right) + \frac{30}{60} \left(\frac{10^2 e^{10\delta_3} 30}{e^{10\delta_3} 30} \right) \\ &= \frac{1}{6}(4) + \frac{2}{6}(25) + \frac{3}{6}(100) = 59.\end{aligned}$$

Exercise 11-20:

A 3-year loan at 10% effective is being repaid with level annual payments at the end of each year.

- (a) Calculate the jump in duration at the time of the first payment.
- (b) Rework (a) at the time of the second payment.
- (c) Compare the answers to (a) and (b) and verbally explain the relationship.

Section 11.6 - Matching Assets and Liabilities

Financial institutions must have the assets available to cover liabilities when they arise. Many types of liabilities are known in advance. It is therefore possible to set up investments, like bonds, so that the inflow of cash from the bonds will match the outflow needed to cover the liabilities due at each point in time. This strategy is called

Example A company has a \$10,000 liability due at the end of year 1 and a \$12,000 liability due at the end of year 2. It can purchase 1-year zero coupon bonds at 8% effective and 2-year zero coupon bonds at 9% effective. What is the cost of implementing an absolute matching strategy today?

$$\frac{\$10,000}{1.08} + \frac{\$12,000}{(1.09)^2} = \$9,259.26 + \$10,100.16 = \$19,359.42.$$

Example:

Suppose the liabilities in the previous example are financed with a 1-year zero coupon bond with a yield rate of 6% and a 2-year 5% annual coupon bond with a yield rate of 7%. What is the cost today?

(a) 2-year bond: At the end of year 2 we need

So $F = 11,428.57$. Also with $\nu = \frac{1}{1.07}$, the price of this 2-year bond is

(b) 1-year bond: At the end of year 1 we need

$$= (11,428.57)(.05) + P_{1yr}(1.06)$$

$$\text{producing } P_{1yr} = 8,894.88$$

(c) Total Cost today = $8,894.88 + 11,015.31 = 19,910.19$.

Section 11.7 - Immunization

Redington immunization is a strategy for portfolio management intended to **make the portfolio immune to small changes in the interest rate i** . Let the return at time t be denoted by

$$R_t = A_t - L_t,$$

the difference between asset amount A_t and liability amount L_t . As before we denote the present value under interest rate i as

$$P(i) = \sum_t v^t R_t.$$

The Taylor series expansion of this function around the value $i = i_0 > 0$, takes the form

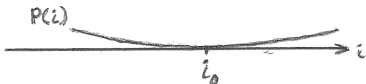
$$P(i_0 + \epsilon) = P(i_0) + \epsilon P'(i_0) + \frac{\epsilon^2}{2} P''(i_0) + \frac{\epsilon^3}{6} P'''(i_0) + \dots$$

The curvature of $P(i)$ at $i = i_0$ is can be approximated by the first three terms.

Redington immunization:

An asset management strategy that chooses assets so that at the current interest rate i_0 ,

When this is possible, it creates a present value function $P(i)$ with a



Under this strategy, **small changes in i can only improve the present value**. A competitive, efficient market makes achieving this condition difficult, but it is a desirable goal. In many settings assets are more amenable to management than are liabilities. Thus the strategy is to make

where $A(i_0)[L(i_0)]$ denotes the present value of the asset [liability] stream at the current interest rate i_0 .

Example:

Suppose you have a liability of L_2 due at $t = 2$, but you can create assets x at $t = 1$ and y at $t = 3$. If i_0 is the current effective annual interest rate, what values should be assigned to x and y to create Redington immunization at $i = i_0$?

Note that

$$P(i) = \frac{x}{(1+i)} - \frac{L_2}{(1+i)^2} + \frac{y}{(1+i)^3},$$

$$P'(i) = -\frac{x}{(1+i)^2} + \frac{2L_2}{(1+i)^3} - \frac{3y}{(1+i)^4}, \quad \text{and}$$

$$P''(i) = \frac{2x}{(1+i)^3} - \frac{6L_2}{(1+i)^4} + \frac{12y}{(1+i)^5}.$$

We seek to find x and y such that

$$P(i_0) = 0, \quad \text{and} \quad P'(i_0) = 0$$

assuming that L_2 and $\nu_0 = \frac{1}{1+i_0}$ are fixed, known values.

Setting $P(i_0) = 0$ produces

$$\nu_0 x + \nu_0^3 y - \nu_0^2 L_2 = 0 \quad \text{or}$$

$$x + \nu_0^2 y - \nu_0 L_2 = 0 \quad (1)$$

Then setting $P'(i_0) = 0$ produces

$$-\nu_0^2 x - 3\nu_0^4 y + 2\nu_0^3 L_2 = 0 \quad \text{or}$$

The simultaneous solution of equations (1) and (2) for x and y yields

$$-3\nu_0^2 y + 2\nu_0 L_2 + \nu_0^2 y - \nu_0 L_2 = 0 \quad \text{or}$$

$$y = \frac{\nu_0 L_2}{2\nu_0^2} = \frac{L_2}{2\nu_0}$$

Moreover, at this solution

$$\begin{aligned} P''(i_0) &= 2 \frac{\nu_0 L_2}{2} \nu_0^3 - 6L_2 \nu_0^4 + 12 \frac{L_2}{2\nu_0} \nu_0^5 \\ &= L_2 \nu_0^4 > 0 \end{aligned}$$

So the conditions of Redington immunization are satisfied.

Specific Example: (using the above setting)

If $L_2 = \$10,000$ and $i_0 = .05$, then Redington immunization is satisfied at $i_0 = .05$ with assets of

$$x = \frac{(\$10,000)}{(1.05)^2} = \$4,761.90 \quad \text{at } t = 1 \text{ and}$$

$$y = \frac{(\$10,000)(1.05)}{2} = \$5,250.00 \quad \text{at } t = 3.$$

Exercise 11-30: A company owes \$100 to be paid at times 2, 4, and 6. The company plans to meet these obligations with an investment program that produces assets of A_1 at time 1 and A_5 at time 5. The current effective rate of interest is 10%.

(a) Determine A_1 and A_5 so that $P(.1) = 0$ and $P'(.1) = 0$.

(b) Does this investment program satisfy the conditions of Redington immunization at $i_0 = .10$?

Section 11.8 - Full Immunization

The previous solution yields a **local minimum** for the function $P(i)$ at interest rate $i = i_0$. In certain continuous growth settings, it may be possible to achieve a **global minimum** for $P(i)$ at $i = i_0$, making the

Setting: Suppose you have continuous compounding with constant force of interest $\delta = \ln(1 + i)$, where i is the effective annual rate of interest. Then

$$a(t) = e^{\delta t} \quad \text{for } t > 0$$

Suppose you have liability L_0 due at time t_0 . Suppose also that you have asset A at time $t = (t_0 - a)$ and asset B at time $t = (t_0 + b)$ where $a > 0$, $b > 0$ and $(t_0 - a) > 0$. Here a and b are both known, but A and B are both unknown.

Now

$$P(\delta) = Ae^{-(t_0-a)\delta} + Be^{-(t_0+b)\delta} - L_0e^{-t_0\delta}$$

and

$$P'(\delta) = -t_0P(\delta) + e^{-t_0\delta}[Aae^{a\delta} - Bbe^{-b\delta}]$$

For a specified force of interest value $\delta = \delta_0$, we seek to find A and B so that **both**

$$P(\delta_0) = 0 \quad \text{and} \quad P'(\delta_0) = 0.$$

So if a and b are known, we get **two linear equations in the two unknowns A and B** , namely:

$$e^{a\delta_0} A + e^{-b\delta_0} B - L_0 = 0 \quad (3)$$

and

These two equations have solutions:

$$A_0 = \frac{\left(\frac{b}{a}\right)L_0}{e^{a\delta_0}\left(1 + \frac{b}{a}\right)}$$

and

$$B_0 = \frac{L_0}{e^{-b\delta_0}\left(1 + \frac{b}{a}\right)}.$$

Using this solution and an arbitrary value of δ , the present value function as a function of δ , is

We now examine the part in the square bracket, namely

$$f(\delta) \equiv [A_0 e^{a\delta} + B_0 e^{-b\delta} - L_0]$$

Note that

$$f'(\delta) = A_0 a e^{a\delta} - B_0 b e^{-b\delta} \quad \text{and}$$

$$f''(\delta) = A_0 a^2 e^{a\delta} + B_0 b^2 e^{-b\delta} > 0 \quad \text{for all } \delta > 0.$$

Therefore, the function $f(\delta)$ is strictly convex. Since $f(\delta_0) = 0$ and $f'(\delta_0) = 0$ (see (3) and (4) on the previous page) the variable value $\delta = \delta_0$ is the unique minimum for $f(\delta)$ among variable values $\delta > 0$. That is, $f(\delta_0) = 0$ and $f(\delta) > 0$, for all positive values of $\delta \neq \delta_0$.

Since

$$P(\delta) = e^{-t_0\delta} f(\delta)$$

It follows that

Therefore the present value function $P(\delta)$ has a unique minimum at $\delta = \delta_0$. At any other force of interest (interest rate) the present value is always an improvement over its value at $\delta = \delta_0$.

Example: (See page 11-33)

At $t_0 = 2$ suppose there is a liability of $L_2 = \$10,000$.

Also let $a = 1$, $b = 1$ and $i_0 = .05$ (or $\delta_0 = \ln(1 + .05)$).

Then

$$A_0 = \frac{L_2}{e^{\delta_0(2)}} = \frac{\nu_0 L_2}{2} = \$4,761.90$$

and

$$B_0 = \frac{L_2}{e^{-\delta_0(2)}} = \frac{L_2}{2\nu_0} = \$5,250.00.$$

With these assets, A_0 at time $t = 1$ and B_0 at time $t = 3$, the present value function has a minimum value of 0 at $\delta_0 = \ln(1 + .05)$ ($i_0 = .05$) and it is greater than zero at **all** other force of interest (interest rate) values.