## Chapter 10 - Term Structure of Interest Rates

## Section 10.2 - Yield Curves

In our analysis of bond coupon payments, for example, we assumed a constant interest rate, $i$, when assessing the present value of the future payments. The formula developed in Chapter 06 gave:

$$
\begin{aligned}
P & =F r a_{n \mid i}+C \nu^{n} \\
& =F r\left(\frac{1-\nu^{n}}{i}\right)+C \nu^{n} .
\end{aligned}
$$

But appropriate interest rates typically vary with the length of the term of investment. A payment of $\$ 100$ five years from today should be assessed with the interest rate associated with a five year zero-coupon bond that is available today. Likewise a payment of $\$ 10010$ years from today should be assessed with the interest rate of a ten year zero-coupon bond that is for sale today. These two interest rates will likely differ.

When we focus on the interest rates of available zero-coupon bonds, the relationship between term length and the effective annual rate of interest is pictured and quantified in a yield curve.

Eff. Int. Rate


## Bond Term

This is a smoothed representation of a normal (typical) yield curse, which is an increasing function of the zero-coupon bond term (length). Indeed, higher interest rates are usually required
to attract investors into longer termed investments. However, at times of high inflation, the Federal Reserve Board will exercise its power to raise short term interest rates in an effort to curb inflation. this produces an inverted yield curve like the one pictured below which shows yield (effective annual interest rate) as a decreasing function of term length. Yield curves can take many shapes including fairly flat curves and ones with bumps.

Eff. Int. Rate


Bond Term

## Section 10.3 - Spot Rates

When assessing the value of a payment (return) $R_{t}>0$ or a deposit $R_{t}<0$, it is appropriate to use the yield rate $s_{t}$ from the yield curve at that particular time $t$. The rate, $s_{t}$, is called the spot rate. It represents the current yield of an investment maturing at the particular point (spot) in time $t$ in the future. The net present value of a sequence of returns $R_{1}, R_{2}, \cdots, R_{n}$ is then

## Example:

Find the present value (price) of a four year annuity immediate in which the first annual payment is $\$ 5,000$ and subsequent annual payments increase by $10 \%$.
Assume the spot rates follow the formula

$$
s_{t}=.02(1.1)^{t-1}+.03
$$

| $t$ | $R_{t}$ | $s_{t}$ | $\left(1+s_{t}\right)^{-t} R_{t}$ |
| :---: | :---: | :---: | :---: |
| 1 | 5,000 | .05 | 4761.90 |
| 2 | 5,500 | .052 | 4969.71 |
| 3 | 6,050 | .0542 | 5164.00 |
| 4 | 6,655 | .05662 | 5339.16 |
|  |  |  | $N P V=\$ 20,234.77$ |

## Exercise

Person A invests $\$ 10,000$ in a 4 -year zero coupon bond. If the rate of inflation is $5 \%$ per annum for the first two years and $4 \%$ per annum for the second two years, find the accumulated value in "today's dollars" of A's investment at the end of four years if the spot rates are given by

$$
s_{t}=.02(1.1)^{t-1}+.03
$$

## Section 10.4 - Relationship with Bond Yield

Spot rates are useful in determining an appropriate price, but an investor wants to determine an overall yield associated with the investment. The differing spot rates will make it difficult for the investor to comprehend the overvalue of the investment. So we seek to produce one rate that is consistent with the net present value of the investment. When purchasing a coupon bond, for example, the spot rates produce a price

$$
P=F r \sum_{t=1}^{n}\left(\frac{1}{1+s_{t}}\right)^{t}+C\left(\frac{1}{1+s_{n}}\right)^{n} .
$$

Using the Law of One Price, we equate this price to a bond with a consistent overall yield of $i$ and solve for $i$. That is, we use the above $P$ and solve for $i$ in the equation

Example: Suppose we assess a 2 -year bond of $\$ 1000$ with $6 \%$ annual coupons. The current spot rates are $5 \%$ for one year and $5.5 \%$ for two years. What is the overall yield of such an investment?

The price determined by the spot rates is

$$
\begin{aligned}
P= & 1000(.06)\left[\left(\frac{1}{1+.05}\right)+\left(\frac{1}{1+.055}\right)^{2}\right]+1000\left(\frac{1}{1+.055}\right)^{2} \\
& =1,009.50 .
\end{aligned}
$$

Set this equal to the current price of a bond with a consistent yield $i$ (written here in terms of $\nu$ ).

$$
1,009.50=
$$

This is a quadratic equation in $\nu$ with solution

$$
\nu=.947997 \quad \text { or } \quad i=.05486 \text {. }
$$

Earlier we specified the spot rate, $s_{t}$, as the yield rate of a zero-coupon bond that matures at future time $t$. But sometimes the desired zero-coupon bonds are not available. So next we explore how to use the prices and features of coupon bonds to determine appropriate spot prices.
Suppose we know the following for $t=1,2, \cdots, n$ :
$P_{t}=$ price of a t-year coupon bond
$F_{t}=$ this bond's face value
$r_{t}=$ this bond's coupon rate and
$C_{t}=$ this bond's redemption value.
Thus for $t=1$, the present value equation produces
which we then solve for the value of $s_{1}$.

Having found the value of $s_{1}$, we then use time $t=2$ setting up the equation

Here everything is known except for $s_{2}$, so we solve it for $s_{2}$.
This iterative method continues. Having found $s_{1}, s_{2}, \cdots, s_{k-1}$, we can then solve

$$
\begin{aligned}
P_{k}=F_{k} r_{k}\left[\left(\frac{1}{1+s_{1}}\right)\right. & \left.+\cdots+\left(\frac{1}{1+s_{k-1}}\right)^{k-1}+\left(\frac{1}{1+s_{k}}\right)^{k}\right] \\
& +C_{k}\left(\frac{1}{1+s_{k}}\right)^{k}
\end{aligned}
$$

for the only unknown quantity $s_{k}$, etc. This iterative solution process is described as a bootstrap method.

Example: The price of a 1 -year $\$ 1000$ bond with a $\$ 80$ coupon payment is $\$ 1000$. While a 2 -year $\$ 1000$ bond with annual coupon payments of $\$ 100$ sells for $\$ 1050$. What are the spot rates for 1 and 2 years?

Using $t=1$,

$$
1000=\frac{80+1000}{1+s_{1}} \quad \text { or } \quad s_{1}=.08
$$

Using $t=2$,

$$
1050=
$$

It follows that

$$
\begin{aligned}
& 1100\left(\nu_{2}\right)^{2}=957.41 \quad \text { or } \\
& \left(\nu_{2}\right)^{2}=.87037, \quad \nu_{2}=.932936 \quad \text { and } \quad s_{2}=.07188 .
\end{aligned}
$$

Some bonds when issued are specified at have at par yield. This means that its yield rate and its coupon rate are the same. Such a bond would sell at par which means, $P=F=C$. Therefore the price equation changes from

$$
P=F r\left[\sum_{t=1}^{n}\left(\frac{1}{1+s_{t}}\right)^{t}\right]+C\left(\frac{1}{1+s_{n}}\right)^{n}
$$

to

$$
F=F i\left[\sum_{t=1}^{n}\left(\frac{1}{1+s_{t}}\right)^{t}\right]+F\left(\frac{1}{1+s_{n}}\right)^{n},
$$

producing

## Example:

Using $s_{t}=.05+(.005) t^{2}$, for $t=0,1$ and 2 , calculate the "at par" yield rate for a two-year bond.

Note that

$$
s_{1}=.055 \quad \text { and } \quad s_{2}=.07
$$

Therefore,

$$
i=\frac{1-(1.07)^{-2}}{(1.055)^{-1}+(1.07)^{-2}}=.069489
$$

## Exercise10-8:

Current term structure is defined by $s_{t}=.06+(.01) t$ for $t=0,1,2,3$
(a) Calculate the at-par yield rate of a two year bond.
(b) Calculate the at-par yield rate of a three-year bond.

## Section 10.5 - Forward Rates

Recall that a spot rate $s_{t}$ represents the interest rate appropriate if you borrowed $\$ \mathrm{R}$ and had to repay it all plus interest at time $t$, i.e. at time $t$ you repay

$$
R\left(1+s_{t}\right)^{t} .
$$

So we might borrow $\$$ R at rate $s_{1}$ for one year or $s_{2}$ if it is due at the end of 2 years, etc. Suppose instead we borrow $\$ R$ for one year and then at the end of the year we borrow all that is due, namely $R\left(1+s_{1}\right)$, for an additional one year period at interest rate $f_{1}$. Then at the end of the second year we would owe

$$
R\left(1+s_{1}\right)\left(1+f_{1}\right) .
$$

For this to be equivalent to a two year loan at an effective annual rate of $s_{2}$, the rate $f_{1}$ must satisfy

$$
\left(1+s_{1}\right)\left(1+f_{1}\right)=\left(1+s_{2}\right)^{2}
$$

Here $f_{1}$ is called a one-year forward rate because it applies to a time period of one year beginning when year one ends.

In general, $f_{n-1}$ is the one-year forward interest rate for money borrowed for one year beginning at the end of year $n-1$. For this interest rate to be consistent with the spot rates for years $\mathrm{n}-1$ and n , it must satisfy

Note also that

$$
f_{0} \equiv s_{1}
$$

It follows that

$$
\begin{aligned}
& \left(1+s_{1}\right)=\left(1+f_{0}\right) \\
& \left(1+s_{2}\right)^{2}=\left(1+s_{1}\right)\left(1+f_{1}\right)=\left(1+f_{0}\right)\left(1+f_{1}\right) \\
& \left(1+s_{3}\right)^{3}=\left(1+s_{2}\right)^{2}\left(1+f_{2}\right)=\left(1+f_{0}\right)\left(1+f_{1}\right)\left(1+f_{2}\right)
\end{aligned}
$$

or

$$
s_{n}=\left[\prod_{t=0}^{n-1}\left(1+f_{t}\right)\right]^{\frac{1}{n}}-1
$$

We see from these relationships that the one-year forward rates can be derived from the spot rates and the spot rates can be derived from the one-year forward rates.

## Example

What are the one-year forward rates for $t=0,1,2,3$ if the spot rates are given by

$$
s_{t}=.05+t(.008) \quad \text { for } t=1,2,3,4
$$

$$
\begin{aligned}
& s_{1}=.058 \quad s_{2}=.066 \quad s_{3}=.074 \quad s_{4}=.082 \quad \text { produce } \\
& f_{0}=.058 \\
& f_{1}=\frac{(1+.066)^{2}}{(1+.058)}-1=.07406 \\
& f_{2}=\frac{(1+.074)^{3}}{(1+.066)^{2}}-1=.09018 \\
& f_{3}=\frac{(1+.082)^{4}}{(1+.074)^{3}}-1=.10636
\end{aligned}
$$

The term forward rate, $f_{t}$, refers to a one-year forward rate as described above. It is also referred to as a t-year deferred one year forward interest rate. This can be generalized to a t-year deferred $m$-year forward rate, $m f_{t}$, which is defined to satisfy
or

$$
{ }_{m} f_{t}=\frac{\left(1+s_{t+m}\right)^{t / m+1}}{\left(1+s_{t}\right)^{t / m}}-1
$$

## Example

Using the setting of the previous example, find the 2-year deferred 2-year forward interest rate.

$$
\begin{aligned}
{ }_{2} f_{2} & =\frac{(1+.082)^{2}}{(1+.066)}-1 \\
& =.09824
\end{aligned}
$$

## Exercise 10-16:

Consider the forward interest rate defined by

$$
f_{k}=.09+.002 k-.002 k^{2} \quad \text { for } \quad k=0,1,2,3,4
$$

(a) Find the 4-year spot rate
(b) Find the 2-year deferred 3 year forward rate.

