Chapter 07 - Yield Rates

Section 7.2 - Discounted Cash Flow Analysis

Suppose an investor makes regular withdrawals and deposits into an investment project. Let R_t denote the return at time $t = 0, 1, \cdots, n$. We assume that the times are evenly spaced. If

$R_t > 0$ then it represents a cash withdrawal to the investor from the project

and if $R_t < 0$ it is a negative withdrawal, i.e. a deposit from the

investor into the project.

$R_t = 0$ is possible.

[We could equivalently describe the situation in terms of $C_t = -R_t$, where $C_t > 0$ is a deposit and $C_t < 0$ is a withdrawal.]

Example 1

Period	Investments into project	Returns from project	Net Cash Flow
0	25,000	0	-25,000
1	10,000	0	-10,000
2	0	2,000	2,000
3	1,000	6,000	5,000
4	0	10,000	10,000
5	0	30,000	30,000
Total	36,000	48,000	12,000

To evaluate the investment project we find the net present value of the returns, i.e.

$$NPV = P(i) = \sum_{t=0}^{n} \nu^t R_t$$

which can be positive or negative depending on the interest rate i.

Example 1 from page:

$$\begin{array}{ll} i = .02 & P(i) = \$8,240.41 \\ i = .06 & P(i) = \$1,882.09 \\ i = .10 & P(i) = -\$3,223.67 \end{array}$$

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The yield rate (also called the internal rate of return (IRR)) is the interest rate i that makes

i.e. this interest rate makes the present value of investments (deposits) equal to the present value of returns (withdrawals).

The solution for this interest rate, i, is the process of finding the appropriate root of a n-degree polynomial. This can be found iteratively using the Newton-Raphson method. It begins with a rough approximation ν_0 and iterates through

until iterations produce insignificant changes. Here

$$
\begin{aligned}
f(\nu) &= \sum_{t=0}^{n} \nu^t R_t \qquad \textit{and} \\
f'(\nu) &= \sum_{t=1}^{n} t\nu^{t-1} R_t
\end{aligned}
$$

Example 1 from three pages earlier:

Use $i_0 = .075$ and thus $\nu_0 = \frac{1}{1.075}$ and it follows that

$$\nu_1 = \frac{1}{1.075} - \frac{-.162093}{151.2194} = .931304$$

and thus $i_1 = .073763$. Continuing the iteration produces

$$\nu_2 = .931304 - \frac{.000285}{151.88339} = .931302$$

and thus $i_2 = .073765$.
We see that the iterative process has basically converged on $i = .0738$.

If $R_1 = R_2 = \cdots = R_n \equiv R$, then

$$P(i) = R_0 + Ra_{\overline{n}|i}.$$

The IRR is found by setting $P(i) = 0$ which produces

which can be solved for i with a financial calculator in the manner used in chapter 03.

Exercise 7-3:

The internal rate of return for an investment in which $C_0 = \$3000$, $C_1 = \$1000$, $R_1 = \$2000$ and $R_2 = \$4000$ can be expressed as $\frac{1}{n}$. Find n.

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Section 7.3 - Uniqueness of the Yield Rate

The solution for the yield rate identifies the ν-values that solve

$$\sum_{t=0}^{n} \nu^t R_t = 0$$

which is a degree n polynomial in ν. Descarte's Rule of Signs says that the number of positive real roots of this equation is at most equal to the number of sign changes in $R_0, R_1, \cdots, R_n$. So as long as this string changes sign one time there is only one solution for ν and hence only one appropriate yield rate i.

Example 1 from previous section:
Here $R_0 = -25000$, $R_1 = -10000$, $R_2 = 2000$, $R_3 = 5000$, $R_4 = 10000$, and $R_0 = 30000$. One sign change means that our previous solution $i = .0738$ is unique.

Example:

Payments of \$500 now and \$550 two years from now are equivalent to a payment of \$1049 one year from now under what interest rate?

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The equation of value is

The roots of this equation are

$$\nu = \frac{1049 \pm \sqrt{(-1049)^2 - 4(500)(550)}}{2(550)}$$

$$= \frac{1049 \pm 20.05}{1100}$$

$\nu_1 \doteq .97184$ $\qquad$ $\nu_2 \doteq .9354318$

$i_1 \doteq .028975$ $\qquad$ $i_2 \doteq .069025$

Which root is the more appropriate?

Exercise 7-5:

Project P requires an investment of \$4000 at time 0. The investment pays \$2000 at time 1 and \$4000 at time 2. Project Q requires an investment of \$x at time 2. The investment pays \$2000 at time 0 and \$4000 at time 1. Using the net present value method at an effective interest rate of 10%, the net present value of the two projects are equal. Calculate x.

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Exercise 7-8:

Payments of \$100 now and \$108.15 two years from now are equivalent to a payment of \$208 one year from now at either rate i or j. Find the absolute difference between the two rates.

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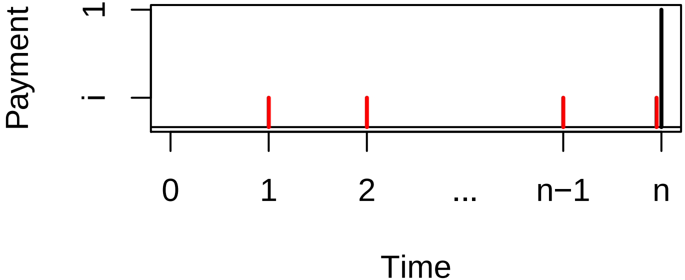
Section 7.4 - Reinvestment Rates

Suppose the situation is such that payments received cannot be reinvested at an interest rate that is equal to that of the original investment project. For example, suppose 1 is invested in an account which pays i per period for each of n periods and then returns the principal of 1 at time $t = n$. Suppose also that the interest earned is reinvested at rate j.



The accumulated value at time $t = n$ is:

$$= 1 + is_{\overline{n}|j} \quad = 1 + \frac{i}{j}((1+j)^n - 1)$$

Note that when $j = i$, this formula becomes the usual compound interest formula :

$$(1+i)^n, \qquad \text{but often } j \text{ is less than } i.$$

Now suppose 1 is invested at the end of each of the n periods. Each deposit earns interest at rate i, but all interest earned is reinvested at rate j. Then the accumulated value at time $t = n$ is:

$$(1 + is_{\overline{n-1}|j}) + (1 + is_{\overline{n-2}|j}) + \cdots + (1 + is_{\overline{1}|j}) + 1$$

$$= n + i \sum_{t=1}^{n-1} s_{\overline{n-t}|j}$$

$$= n + i\sum_{t=1}^{n-1}\frac{((1+j)^t - 1)}{j}$$

$$= n + \frac{i}{j}\Big(\frac{(1+j)[(1+j)^{n-1} - 1]}{(1+j) - 1} - (n-1)\Big) \qquad SGS$$

$$= n + \frac{i}{j}\Big(\frac{(1+j)^n - 1 - j}{j} - (n-1)\Big)$$

Example

A $1000 bond with 5% coupon rate payable semiannually is redeemable in 5 years at face value. The price is set so that the investor expects a 6% yield convertible semiannually, but the payments can only be reinvested at 4% convertible semiannually. What is the investor's actual yield rate from this investment?

The price of the bond is :

$$\begin{aligned} P &= C + C(g-i)a_{\overline{n}|i} \\ &= 1000 + 1000(.025 - .03)a_{\overline{10}|.03} \\ &= 957.35 \end{aligned}$$

At $t = 10$ the \$25 coupon payments plus the redemption value accumulate to

$$1000 + 25s_{\overline{10}|.02}$$

$$1000 + 25\frac{[(1.02)^{10} - 1]}{.02} \qquad = 1273.74.$$

So the actual semiannual yield i' satisfies

The annual effective interest rate is found by

$$(1+i) = (1 + .028996)^2 \qquad or \qquad i = .05877$$

Exercise 7-12:

A loan of \$10,000 is being repaid with payments of \$1000 at the end of each year for 20 years. If each payment is immediately reinvested at 5% effective, find the effective annual rate of interest earned over the 20-year period.

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Section 7.5 - Interest Measurement of a Fund

The purpose of this section is to describe a yield rate for an investment fund over a single period (typically one year). Let

A = amount in the fund at the beginning of the period

B = amount in the fund a the end of the period

I = amount of interest earned during the period

C_t = net amount contributed to the fund at time t, $0 \leq t \leq 1$.

$C_t > 0$ denotes a deposit

$C_t < 0$ denotes a withdrawal

C = total amount contributed during the period

$$C = \sum_t C_t$$

It follows from these descriptions that

For the interest to be consistent with a yield rate of i,

$$\mathrm{I} = iA + \sum_t C_t(\text{interest rate from t to 1})$$

where the interest rate from t to 1 is

$$\begin{aligned} {}_{1-t}i_t &= (1+i)^{1-t} - 1 \quad \text{under compound interest} \\ &= i(1-t) \qquad \text{under simple interest} \end{aligned}$$

Under compound interest,

$$B = (1+i)A + \sum_t C_t(1+i)^{1-t} \qquad \text{or}$$

$$A = \nu B - \sum_t C_t \nu^t$$

which requires an iterative process to solve for ν and thus i.

Under simple interest,

$$\mathrm{I} = iA + \sum_t C_t i(1-t) \qquad \text{or}$$

Here the denominator $A + \sum_t C_t(1-t)$ is called the exposure associated with i. A simple approximation to this i can be obtained by assuming all contributions occurred at time $t = 1/2$, in which case

$$\begin{aligned} i &= \frac{\mathrm{I}}{A + .5C} \\ &= \frac{2\mathrm{I}}{A + B - \mathrm{I}} \end{aligned}$$

Example:

A fund begins the year with a balance of \$1000. A deposit of \$500 is made at the end of the first quarter and a withdrawal of \$200 is made at the end of the 2nd and 3rd quarters. What is the yield if the fund finishes the year with a balance of \$1220? Use simple interest.

$$A = 1000 \qquad B = 1220 \qquad C = 100.$$

Note that I = 120 and

$$\sum_t C_t(1-t) = 500(3/4) + (-200)(1/2) + (-200)(1/4) = 225.$$

It follows that

The yield computation method described above is said to be dollar weighted, because the dollar contributions (or withdrawals) of the investor play a major role in determining the yield. Consider the previous example , but with $C_1 = -400$, $C_2 = 0$, and $C_3 = 500$ at the ends of the 1st, 2nd and 3rd quarters, respectively. Use A and B as the same as above and note that C and I are also the same. But

$$\sum_t C_t(1-t) = -400(3/4) + 500(1/4) = -175.$$

and therefore

$$i = \frac{120}{1000 - 175} = .14545.$$

Thus yield is highly dependent on the timing and contribution amounts of the investor.

Exercise 7-20:

An investment account earning 6% is established with an balance at the beginning of the year of \$10,000. There are new deposits of \$1800 made at the end of two months and another \$900 made at the end of eight months. In addition, there is a withdrawal of \$k made at the end of six months. The fund balance at the end of the year is \$10,636. Determine k to the nearest dollar using simple interest.

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Section 7.6 - Time Weighted Rates of Interest

When assessing the performance of an investor, dollar weighting is quite appropriate. But when assessing the performance of the fund itself and/or the fund administrators, this strong dependence on the timing and contribution amounts of the investor is less appropriate.

Suppose contributions are made at times t_k for $k = 1, 2, \cdots, m-1$ at times $t_0 \equiv 0 < t_1 < t_2 < \cdots < t_{m-1} < t_m \equiv 1$. Let

$$B'_k = \text{ the fund balance just before the contribution at time } t_k$$

$$C'_k = \text{ the contribution amount made at time } t_k$$

$$C'_k > 0 \text{ indicating a deposit and } C'_k < 0, \text{ a withdrawal.}$$

The interest rate, j_k, achieved during the time period (t_{k-1}, t_k) satisfies

$$(B'_{k-1} + C'_{k-1})(1 + j_k) = B'_k \qquad \textit{or}$$

$$(1 + j_k) = \frac{B'_k}{B'_{k-1} + C'_{k-1}}$$

where we recognize that $(1 + j_k)$ can be greater than or less than 1. The accumulated value over $(0, 1)$, set equal to $(1 + i)$ produces

The value i which satisfies this equation is the time-weighted yield of the fund and is more reflective of the performance of the fund itself and less dependent on the timing and contribution amounts of the investor.

Example (continuation from section 7.5)

Earlier we found the dollar weighted yield to be $i = .09796$. We will now compute the time-weighted yield. For this we need additional information, namely the values of B'_1 =\$1020, B'_2=\$1555, and B'_3=\$1482. Those values and the ingredients of the computations are shown in the table below.

k	B'_k	C'_k	$(1 + j_k)$
0	0	1000	
1	1020	500	1.0200
2	1555	-200	1.0230
3	1482	-200	1.0937
4	1220	0	0.9516

Here, for example,

$$(1 + j_2) = \frac{1555}{1020 + 500} = 1.0230$$

and

$$(1 + j_4) = \frac{1220}{1482 - 200} = 0.9516.$$

To compute the yield we form:

Therefore, the time weighted yield is

$$i = .0860$$

Exercise 7-26:

You invest \$2000 at time $t = 0$ and an additional \$1000 at time $t = 1/2$. At time $t = 1$ you have \$3200 in your account. Find the amount that would have to be in your account at time $t = 1/2$, if the time weighted rate of return over the year is exactly .02 higher than the dollar weighted rate of return. Assume simple interest in calculating the dollar weighted return.

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Exercise 7-28:

An investor deposits 50 in an investment account on January 1. The following summarizes the activity in the account during the year:

Date	Value just before deposit	Deposit
March 15	40	20
June 1	80	80
October 1	175	75

On June 30 the value of the account is \$157.50. On December 31 the value of the account is x. Using the time weighted method, the equivalent annual yield during the first 6 months is equal to the time-weighted annual yield during the entire one year period. Calculate x.

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Section 7.7 Portfolio Method

An investment fund is typically created to serve a number of individual persons or companies. When it comes to crediting interest to the individual accounts, the fund can do this in one of several ways.

Many funds are set up so that deposits to the fund buy shares in the fund. Interest from the fund's performance for a given year is distributed to individual accounts in proportion to the number of shares held in the account during that year. This is called the portfolio method of assigning a yield for the individual accounts. With this method, the same interest rate is applied to the total amount accrued in each account in a particular year. This is a allocation method that is relatively simple to perform and to explain to fund members.

Some funds use a more complicated method designed to attract new money into the fund during times of rising interest (improving fund performance). One such method is called the investment year method. This method is described in our textbook, but we will not describe it here because it has been dropped from the FM exam.