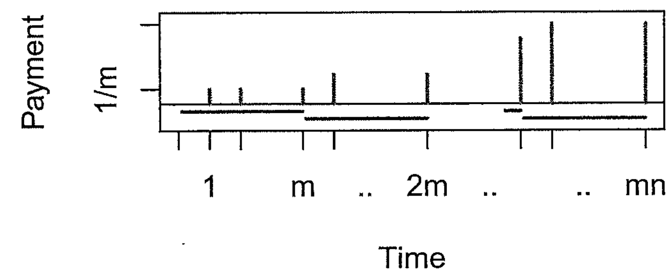
A person will receive \$2, 000 at the beginning of each month of the first year. Every year thereafter the quarterly payments will increase by 10%, but they always remain the same within a year. The payments cease after 20 years. If the effective annual interest rate is 6%, find the present value of this series of payments.

Section 4.8 - More General Varying Annuities

Consider settings in which the interest conversion periods and the payment periods do not coincide.

For example, suppose we have m payments within each interest conversion period and payments varying in arithmetic progression over interest conversion periods. Let



$m =$ (number of payments per interest conversion period)

$n =$ (number of interest conversion periods)

$i =$ (effective interest rate per conversion period)

$\frac{k}{m} =$ (amount of each payment in the k^{th} conversion period)

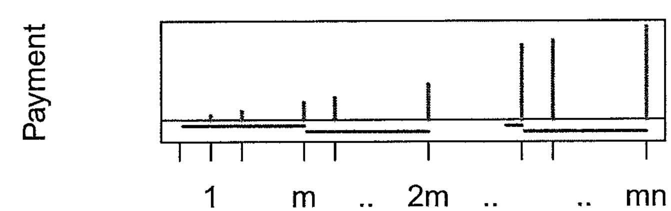
The present value of these payments is:

$$(\mathrm{I}a)^{(m)}_{\overline{n}|} = \frac{1}{m}\Big[(\nu^{\frac{1}{m}} + \nu^{\frac{2}{m}} + \cdots + \nu^{\frac{m}{m}}) + 2(\nu^{\frac{m+1}{m}} + \nu^{\frac{m+2}{m}} + \cdots + \nu^{\frac{2m}{m}})$$
$$+ \cdots + n(\nu^{\frac{m(n-1)+1}{m}} + \nu^{\frac{m(n-1)+2}{m}} + \cdots + \nu^{\frac{nm}{m}})\Big]$$
$$= \frac{1}{m}(\nu^{\frac{1}{m}} + \nu^{\frac{2}{m}} + \cdots + \nu^{\frac{m}{m}})[1 + 2\nu + \cdots + n\nu^{n-1}]$$
$$= \frac{1}{m}\frac{\nu^{\frac{1}{m}}(1-\nu^{\frac{m}{m}})}{(1-\nu^{\frac{1}{m}})}\nu^{-1}(\mathrm{I}a)_{\overline{n}|}$$

$$= \frac{1}{m}\frac{(1-\nu)(\ddot{a}_{\overline{n}|} - n\nu^n)}{(\nu^{-\frac{1}{m}} - 1)(i\nu)}$$

$$\boxed{= \frac{\ddot{a}_{\overline{n}|} - n\nu^n}{i^{(m)}}}$$

Suppose every payment increases in arithmetic progression with a payment amount of $\frac{k}{m^2}$ made at time $t = k$.



The present value of these payments is:

$$(\mathrm{I}^{(m)}a)_{\overline{n}|}^{(m)} = \frac{\ddot{a}_{\overline{n}|}^{(m)} - n\nu^n}{i^{(m)}}.$$

Exercise 4-34

Suppose a deposit of $1000 is made on the first of the months of January, February and March, $1,200 at the beginning of each month in the second quarter, $1,400 at the beginning of each month in the third quarter and $1,600 each month in the fourth quarter. If the account has a nominal 8% rate of interest compounded quarterly, what is the balance at the end of the year?

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$n = 4 \qquad m = 3 \qquad i = .08/4 = .02 \qquad \nu = (1.02)^{-1}$

$$i^{(3)} = 3[(1.02)^{\frac{1}{3}} - 1] = .019868$$

The present value at $t = -1$ is

$$\boxed{800(3)a_{\overline{4}|.02}^{(3)} + 200(3)(Ia)_{\overline{4}|.02}^{(3)}}$$

4-41

$$= 2400\left(\frac{1-(1.02)^{-4}}{.019868}\right) + 600\left(\frac{\frac{1-(1.02)^{-4}}{1-(1.02)^{-1}} - 4(1.02)^{-4}}{.019868}\right)$$

$$= 9,199.20 + 5,692.58 = \$14,891.78$$

At $t = 0$ the value is

$$(1.02)^{\frac{1}{3}}(14,891.78) = \$14,990.40$$

and its value at $t = 12$ is

$$(1.02)^{\frac{13}{3}}(14,891.78) = \$16,226.10$$

which is the balance at the end of the year.

4-42

A perpetuity provides payments every six-months starting today. The first payment is 1 and each payment is 3% greater than the immediately preceding payment. Find the present value of the perpetuity if the effective rate of interest is 8% per annum.

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4-43

Section 4.9 - Continuous Annuities with Varying Payments and/or Force of Interest

The accumulation function $a(t)$ and its reciprocal, the discount function $d(t) = \frac{1}{a(t)}$ play fundamental roles in the assessment of a series of payments when viewed from some specific time.

Consider a series of payments in which p_{t_j} denotes a payment made at time $t = t_j$ for $j = 1, 2, \cdots, n$. The present value of this series of payments is

$$PV = \sum_{j=1}^{n} p_{t_j}\frac{1}{a(t_j)} = a(0)PV \qquad \text{because } a(0) = 1.$$

Likewise consider the future value of this sequence at the final payment, i.e. at $t = t_n$.

$$FV = \sum_{j=1}^{n} p_{t_j}\frac{a(t_n)}{a(t_j)} = a(t_n)\sum_{j=1}^{n} p_{t_j}\frac{1}{a(t_j)} = a(t_n)PV.$$

4-44