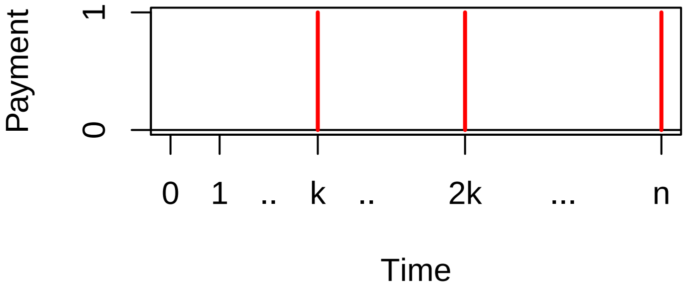
Chapter 04 - More General Annuities

Section 4.3 - Annuities Payable Less Frequently Than Interest Conversion



$k =$ interest conversion periods before each payment

$n =$ total number of interest conversion periods

$n/k =$ total number of payments (positive integer)

$i =$ rate of interest in each conversion period .

General Method

Example: Payments of \$500 are made at the end of each year for 10 years. Interest has a nominal rate of 8%, convertible quarterly.

(a) What is the present value of these future payments?

$i^{(4)} = .08 \qquad i^{(4)}/4 = .02$

$(1 + .02)^4 = 1.08243216$

Therefore 8.243216% is the annual effective interest rate

$$\text{Answer} = 500a_{\overline{10}|.08243216} = \$3,318.54$$

(b) What is the accumulated value of these payments at the end of 10 years?

$$\text{Answer} = 500s_{\overline{10}|.08243216} = \$7,327.48.$$

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Formula Method for Annuity-Immediate

Now view this setting as n periods with spaced payments. The present value of these n/k payments is

$$PV_n = \nu^k + \nu^{2k} + \nu^{3k} + \cdots + \nu^{(n/k)k} \qquad \textit{where} \quad \nu = \frac{1}{1+i}$$

$$= \frac{\nu^k(1-(\nu^k)^{(n/k)})}{1-\nu^k} \qquad \text{by SGS}$$

The accumulated value at time $t = n$ is

$$(1+i)^n \frac{a_{\overline{n}|i}}{s_{\overline{k}|i}} \quad = \quad \frac{s_{\overline{n}|i}}{s_{\overline{k}|i}}$$

Both of the above formulas are annuity-immediate formulas because the payments are at the end of each payment period which is k interest periods long.

Example: (previous in this section)

$$i = .02 \qquad k = 4 \qquad n = 40$$

$$\text{Accumulated Value } = 500 \frac{s_{\overline{40}|.02}}{s_{\overline{4}|.02}} = \$7,327.48$$

Formula Method for Annuity-due:

Present Value:

$$1 + \nu^k + \nu^{2k} + \nu^{3k} + \cdots + \nu^{n-k}$$

$$= \frac{\left(1 - (\nu^k)^{(n/k)}\right)}{1 - \nu^k} \qquad \text{by SGS}$$

Accumulated Value at time $t = n$ is:

$$(1+i)^n \frac{a_{\overline{n}|i}}{a_{\overline{k}|i}} = \frac{s_{\overline{n}|i}}{a_{\overline{k}|i}} = \frac{\ddot{s}_{\overline{n}|i}}{\ddot{a}_{\overline{k}|i}}$$

Both of the above formulas are annuity-due formulas because the payments are at the beginning of each payment period which is k interest periods long.

Perpetuities:

Annuity-immediate with payments less frequent than interest conversion

$$\text{Present Value} = \lim_{n\to\infty} \frac{(1-\nu^n)}{(1+i)^k - 1}$$

Annuity-due with payments less frequent than interest conversion

$$\text{Present Value} = \lim_{n\to\infty} \frac{(1-\nu^n)}{1-\nu^k}$$

$$= \frac{1}{i(1-\nu^k)/i} \quad = \quad \frac{1}{i a_{\overline{k|}i}}$$

Exercise 4-8:

The present value of a perpetuity paying 1 at the end of every 3 years is $\frac{125}{91}$. Find i.

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$$\frac{125}{91} = \frac{1}{i s_{\overline{3}|i}} = \frac{1}{(1+i)^3 - 1}$$

So

$$(1+i)^3 = \frac{91}{125} + 1 = \frac{216}{125}$$

$$1 + i = \frac{6}{5}$$

$$i = .20$$

Exercise 4-4:

An annuity-immediate that pays \$400 quarterly for the next 10 years costs \$10,000. Calculate the nominal interest rate convertible monthly earned by this investment.

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Exercise 4-7:

Find an expression for the present value of an annuity-due of \$600 per annum payable semiannually for 10 years, if $d^{(12)} = .09$.

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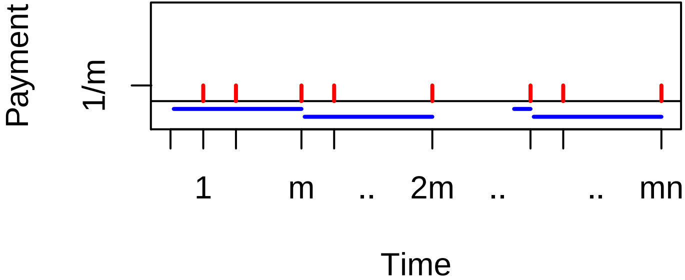
Section 4.4 - Annuities Payable More Frequently Than Interest Conversion



$m =$ number of payments in each interest conversion period

$1/m =$ amount of each payment

$n =$ total number of interest periods

$i =$ effective rate of interest for each conversion period .

The annuity-immediate present value at time $t = 0$ for all payments is

$$a_{\overline{n}|}^{(m)} = \frac{1}{m}\left[\nu^{\frac{1}{m}} + \nu^{\frac{2}{m}} + \cdots + \nu^{\frac{m}{m}} + \cdots + \nu^{\frac{mn}{m}}\right]$$

$$= \frac{1}{m}\left[\frac{1-\nu^n}{\left((1+i)^{\frac{1}{m}} - 1\right)}\right] \quad = \quad \frac{1-\nu^n}{i^{(m)}} \quad = \quad \frac{i a_{\overline{n}|i}}{i^{(m)}}$$

Here $\quad i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right] \quad$ because

$$(1+i) = \left(1 + \frac{i^{(m)}}{m}\right)^m.$$

That is, $i^{(m)}$ is the nominal period interest rate and i is the effective period interest rate when each interest period is converted $m^{th}ly$.

Once we know the present value at time $t = 0$, the accumulated value at the end of the n^{th} conversion period (i.e. at time $t = mn$) is

$$s_{\overline{n}|}^{(m)} = (1+i)^n a_{\overline{n}|}^{(m)}$$

$$= \frac{(1+i)^n - 1}{i^{(m)}} \quad = \quad \frac{i s_{\overline{n}|i}}{i^{(m)}}$$

Example: Payments of \$500 are made at the end of each month for 10 years. Interest is set at 6% (APR) convertible quarterly. What is the accumulated value at the last payment.

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The effective interest rate per quarter is $\frac{.06}{4} = .015$. The quarters are the interest conversion periods. So $n = 4(10) = 40$, $m = 3$ and $i = .015$. Note that

and the accumulated value at the last payment is :

$$(500)(3)s_{\overline{40}|}^{(3)} = \frac{500(3)[(1+.015)^{40} - 1]}{.0149256} = \$81,807.61.$$

The annuity-due present value at time $t = 0$ for all payments is

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - \nu^n}{d^{(m)}} = \frac{i a_{\overline{n}|i}}{d^{(m)}}.$$

The annuity-due accumulated value at the end of the n^{th} conversion period (i.e. at time $t = mn$) is

$$\ddot{s}_{\overline{n}|}^{(m)} = \frac{(1+i)^n - 1}{d^{(m)}} = \frac{i s_{\overline{n}|i}}{d^{(m)}}$$

Here d is the effective rate of discount per interest period and $d^{(m)}$ is the nominal rate of discount per interest period when convertible m^{th}ly in each period.

It easily follows that the extension to an annuity-immediate perpetuity produces a present value of

and the extension to an annuity-due perpetuity produces

$$\ddot{a}^{(m)}_{\overline{\infty}|} = \frac{1}{d^{(m)}}$$

Exercise 4-14: Find i when

$$3a^{(2)}_{\overline{n}|} \quad = \quad 2a^{(2)}_{\overline{2n}|} \quad = \quad 45s^{(2)}_{\overline{1}|}.$$

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These equalities produce

$$\frac{3(1-\nu^n)}{i^{(2)}} = \frac{2(1-\nu^{2n})}{i^{(2)}} = \frac{45i}{i^{(2)}}$$

Using the first and third of these terms produces

$$3(1-\nu^n) = 45i \qquad \text{which implies} \qquad \nu^n = 1 - 15i.$$

Now using the first and second terms above yields

$$3(1-\nu^n) = 2(1-\nu^{2n}). \qquad \text{Setting } x = \nu^n \text{ this becomes}$$

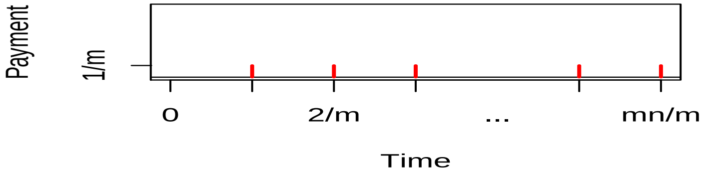
Therefore,

$$x = \nu^n = \frac{1}{2} = 1 - 15i \qquad \text{which implies}$$

$$15i = \frac{1}{2} \qquad \text{or} \qquad i = \frac{1}{30}.$$

Section 4.5 - Continuous Payment Annuities

Consider a m^{th}ly annuity-immediate paying a total of 1 annually over n years.



Examine the payments made in the interval from t to $t + \Delta$, where $0 < t < t + \Delta < n$.

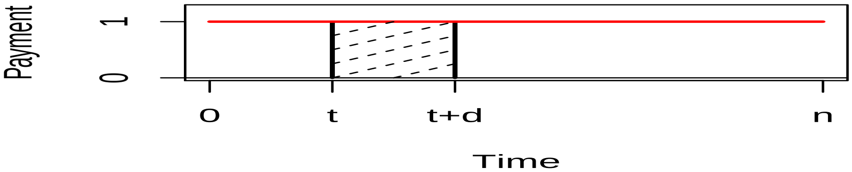
The total of the payments between t and $t + \Delta$ is

$$\sum \frac{1}{m} = \frac{(\# \text{ of } m^{th}\text{ly interval ends between t and t+}\Delta)}{m}$$

When m is very large, this total payment is approximately

So when m is very large, it is (approximately) as though the payment of 1 made each year is smeared evenly over that year and where the payment total in any interval is the area under this line above the interval.

Continuous Model:



When finding the present value of all payments, we must discount the payment made at time t by the factor ν^t. Thus the present value of all payments becomes

$$\begin{aligned}\bar{a}_{\overline{n}|} &= \int_0^n 1 \cdot \nu^t dt \\ &= \frac{\nu^t}{\ln(\nu)}\Big|_0^n \quad = \quad \frac{1}{\ln(\nu)}(\nu^n - 1) \qquad \text{and therefore}\end{aligned}$$

Here the annual effective interest rate i is fixed and $\delta = \ln(1+i)$ is the force of interest. Note that

$$\lim_{m\to\infty} a_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \frac{1-\nu^n}{i^{(m)}} = \frac{1-\nu^n}{\delta} = \bar{a}_{\overline{n}|}$$

because $\lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} d^{(m)} = \delta$. With continuous payments, the distinction between an annuity-immediate and an annuity-due is moot, that is

$$\bar{a}_{\overline{n}|} = \lim_{m\to\infty} a_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \ddot{a}_{\overline{n}|}^{(m)}.$$

It likewise follows that

$$\bar{s}_{\overline{n}|} = \lim_{m\to\infty} s_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \ddot{s}_{\overline{n}|}^{(m)}$$

Finally, since $\delta = \ln(1+i)$ implies that $\nu = e^{-\delta}$ and $(1+i) = e^{\delta}$, it follows that

$$\bar{a}_{\overline{n}|} = \frac{(1-\nu^n)}{\delta} = \frac{1-e^{-n\delta}}{\delta} \quad \text{and}$$

$$\bar{s}_{\overline{n}|} = \frac{[(1+i)^n - 1]}{\delta} = \frac{e^{n\delta}-1}{\delta},$$

where in these expressions n does not have to be an integer.

Exercise 4-18:

If $\bar{a}_{\overline{n}|} = 4$ and $\bar{s}_{\overline{n}|} = 12$, find δ.

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$$4 = \bar{a}_{\overline{n}|} = \frac{1 - e^{-n\delta}}{\delta} \qquad \text{implies} \qquad e^{-n\delta} = 1 - 4\delta$$

$$12 = \bar{s}_{\overline{n}|} = \frac{e^{n\delta} - 1}{\delta} \qquad \text{implies} \qquad e^{n\delta} = 12\delta + 1$$

Putting these two expressions together produces

$$\text{or} \qquad 48\delta^2 - 8\delta = 0.$$

This yields

$$\delta = \frac{8}{48} = .16\bar{6}.$$

Exercise 4-20:

Find the value of t, $0 < t < 1$, such that 1 paid at time t has the same present value as 1 paid continuously between time 0 and 1.

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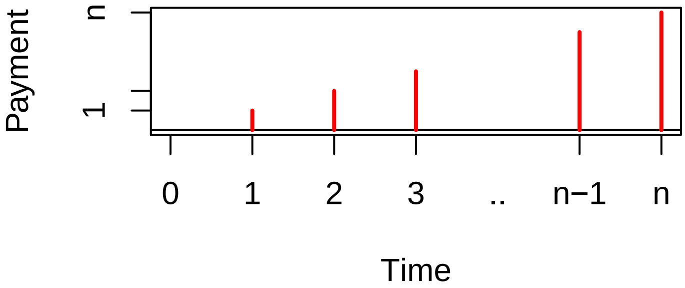
Section 4.6 - Payments in Arithmetic Progression

Suppose an annuity pays k at the end of period k for $k = 1, 2, \cdots, n$.



The present value of this annuity with arithmetic increasing payments is

$$(Ia)_{\overline{n}|} = \nu + 2\nu^2 + 3\nu^3 + \cdots + (n-1)\nu^{n-1} + n\nu^n.$$

Note that

$$\frac{(\mathrm{I}a)_{\overline{n}|}}{\nu} = 1 + 2\nu + 3\nu^2 + \cdots + n\nu^{n-1} \qquad \text{and thus}$$

$$\frac{(\mathrm{I}a)_{\overline{n}|}}{\nu} - (\mathrm{I}a)_{\overline{n}|} = 1 + \nu + \nu^2 + \nu^3 + \cdots + \nu^{n-1} - n\nu^n$$

$$= \ddot{a}_{\overline{n}|} - n\nu^n.$$

It follows that

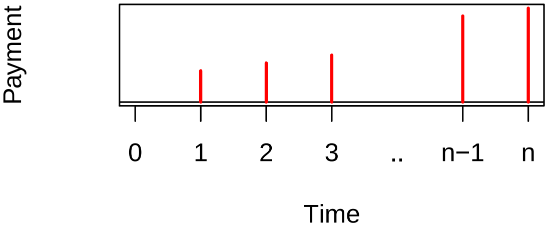
$$\left(\frac{1-\nu}{\nu}\right)(\mathrm{I}a)_{\overline{n}|} = i(\mathrm{I}a)_{\overline{n}|} = \ddot{a}_{\overline{n}|} - n\nu^n \qquad \text{and thus}$$

The accumulated value at time $t = n$ is:

$$(\mathrm{I}s)_{\overline{n}|} = (1+i)^n(\mathrm{I}a)_{\overline{n}|}$$

$$= \frac{\ddot{s}_{\overline{n}|} - n}{i} = \frac{(1+i)s_{\overline{n}|} - n}{i}.$$

Consider a annuity-immediate with general arithmetic progression payment amounts: P at time $t = 1$, $P + Q$ at time $t = 2$, $P + 2Q$ at time $t = 3$, $\cdots$, and $P + (n-1)Q$ at time $t = n$.



The present value at $t = 0$ is

and the accumulated value at $t = n$ is

$$(1+i)^n[\text{present value}] = (1+i)^n\Big[Pa_{\overline{n}|} + Q\Big(\frac{a_{\overline{n}|} - n\nu^n}{i}\Big)\Big]$$

$$= Ps_{\overline{n}|} + Q\Big(\frac{s_{\overline{n}|} - n}{i}\Big)$$

Next examine a decreasing annuity-immediate with a payment of $n + 1 - k$ at time $t = k$, for $k = 1, 2, \cdots, n$. This is a special case of the previous formula with $P = n$ and $Q = -1$.
The present value becomes

$$(Da)_{\overline{n}|} = na_{\overline{n}|} - \Big(\frac{a_{\overline{n}|} - n\nu^n}{i}\Big)$$

and the accumulated value is

$$(Ds)_{\overline{n}|} = (1+i)^n(Da)_{\overline{n}|} = ns_{\overline{n}|} - \Big(\frac{s_{\overline{n}|} - n}{i}\Big).$$

For a perpetuity-immediate with general arithmetic progression payment amounts, the present value is

as long as $P > 0$ and $Q > 0$ because

$$\lim_{n\to\infty} a_{\overline{n}|} = \frac{1}{i} \qquad \text{and} \qquad \lim_{n\to\infty} n\nu^n = 0.$$

Exercise 4-24: Find the present value of a perpetuity that pays 1 at the end of the first year, 2 at the end of the second year, increasing until a a payment of n at the end of the n^{th} year and thereafter payments are level at n per year forever.

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$$(\mathrm{I}a)_{\overline{n}|} + \frac{n\nu^n}{i} = \frac{\nu^{-1}a_{\overline{n}|} - n\nu^n + n\nu^n}{i} = \frac{a_{\overline{n}|}}{d}.$$

Exercise 4-27:

An annuity-immediate has semiannual payments of

$$800 \qquad 750 \qquad 700 \qquad \cdots \qquad 350$$

with $i^{(2)} = .16$. If $a_{\overline{10}|.08}$ = A, find the present value of the annuity in terms of A.

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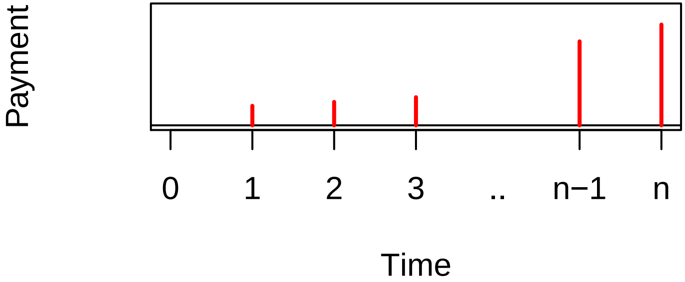
Section 4.7 - Payments in Geometric Progression

Suppose an annuity-immediate pays $(1+k)^{k-1}$ at the end of period k for $k = 1, 2, \cdots, n$.



The present value of these payments is

$$\nu + (1+k)\nu^2 + (1+k)^2\nu^3 + \cdots + (1+k)^{n-1}\nu^n$$

$$= \nu \left[\frac{1 - \left((1+k)\nu\right)^n}{1 - (1+k)\nu} \right]$$

$$= \frac{1}{1+i} \left[\frac{1 - \left(\frac{1+k}{1+i}\right)^n}{1 - \left(\frac{1+k}{1+i}\right)} \right]$$

The accumulated value at $t = n$ is:

$$(1+i)^n[\text{present value}] = \begin{cases} \frac{(1+i)^n - (1+k)^n}{(i-k)} & \text{if } k \neq i \\ n(1+i)^{n-1} & \text{if } k = i \end{cases}$$

When dealing with an annuity-due, the present value and the accumulated value are obtained by multiplying the respective expressions for an annuity-immediate by $(1+i)$.

The present value of a perpetuity with geometrically changing payments only converges to a finite value when $k < i$, in which case it is

$$\left(\frac{1+i}{i-k}\right) \quad \text{for a Perpetuity-Due.}$$

Example: A perpetuity-due pays \$1000 for the first year and payments increase by 3% for each subsequent year until the 20^{th} payment. After that the payments are the same as the 20^{th}. Find the present value if the effective annual interest rate is 5%.

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$$1000(1+.05)\left[\frac{1-\left(\frac{1.03}{1.05}\right)^{20}}{.05-.03}\right]+\frac{1000(1.03)^{19}}{(1.05)^{19}(.05)}$$

$$= 16,763.02 + 13,878.44 = \$30,641.46$$

Exercise 4-32:

An employee age 40 earns $40,000 per year and expects to receive 3% annual raises at the end of each year for the next 25 years. The employee contributes 4% of annual salary at the beginning of each year for the next 25 years into a retirement plan. How much will be available for retirement at age 65 if the fund earns a 5% effective annual rate of interest?

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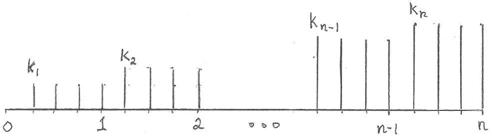
Section 4.7.5 - Varying Blocks of Equal Payments

Suppose each period consists of m equal sub-periods and for the sub-periods within a given period the payments are equal. But the payment amount changes from period to period either in arithmetic or geometric progression.

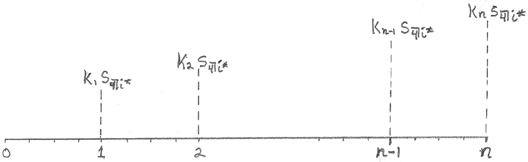
Case A:

During the first period each sub-period payment is k_1, during the second period each sub-period payment is $k_2, \cdots$, and during the n^{th} period each sub-period payment is k_n.



To find the present value of these payments, we first accumulate the payments within each period to the end of their period using the

applicable for each sub-period. This produces an accumulation of at the end of the j^{th} period for $j = 1, 2, \cdots, n$.



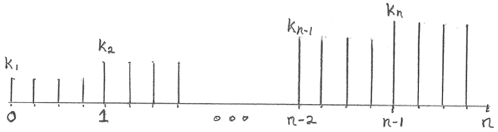
The present value is obtained by including the common factor $s_{\overline{m}|i^*}$ and using either the methods of section 4.6, if the k_m's vary in arithmetic progression, or the methods of section 4.7, if the k_m's vary in geometric progression. These computations would be based on i, the effective interest rate of each complete period.

Example:

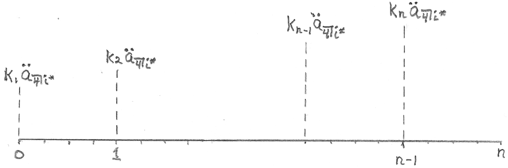
A person will receive \$1, 000 at the end of each quarter of the first year. Every year thereafter the quarterly payments will increase by \$500. During the 11^{th} year (the final year) the payments will be \$6,000 at the end of each quarter. If the effective annual interest rate is 5%, find the present value of this series of payments.

Case B:

During the first period each sub-period payment is k_1, during the second period each sub-period payment is k_2, $\cdots$, and during the n^{th} period each sub-period payment is k_n.



To find the present value of these payments, we first discount the payments within each period to the beginning of their period using the applicable for each sub-period. This produces a present value of at the beginning of the j^{th} period for $j = 1, 2, \cdots, n$.



The present value is obtained by including the common factor $\ddot{a}_{\overline{m}|i^*}$ and using either the methods of section 4.6, if the k_m's vary in arithmetic progression, or the methods of section 4.7, if the k_m's vary in geometric progression. These computations would be based on i, the effective interest rate of each complete period.

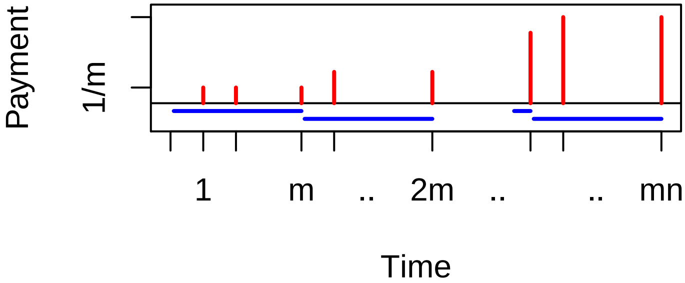
Example:

A person will receive \$2, 000 at the beginning of each month of the first year. Every year thereafter the quarterly payments will increase by 10%, but they always remain the same within a year. The payments cease after 20 years. If the effective annual interest rate is 6%, find the present value of this series of payments.

Section 4.8 - More General Varying Annuities

Consider settings in which the interest conversion periods and the payment periods do not coincide.

For example, suppose we have m payments within each interest conversion period and payments varying in arithmetic progression over interest conversion periods. Let



$m =$ (number of payments per interest conversion period)

$n =$ (number of interest conversion periods)

$i =$ (effective interest rate per conversion period)

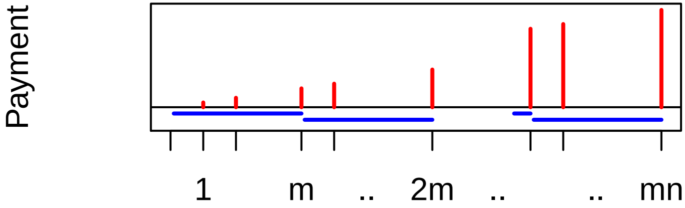
$\frac{k}{m} =$ (amount of each payment in the k^{th} conversion period)

The present value of these payments is:

$$
\begin{aligned}
(\mathrm{I}a)^{(m)}_{\overline{n}|} &= \frac{1}{m}\Big[(\nu^{\frac{1}{m}} + \nu^{\frac{2}{m}} + \cdots + \nu^{\frac{m}{m}}) + 2(\nu^{\frac{m+1}{m}} + \nu^{\frac{m+2}{m}} + \cdots + \nu^{\frac{2m}{m}}) \\
&\qquad + \cdots + n(\nu^{\frac{m(n-1)+1}{m}} + \nu^{\frac{m(n-1)+2}{m}} + \cdots + \nu^{\frac{nm}{m}})\Big] \\
&= \frac{1}{m}(\nu^{\frac{1}{m}} + \nu^{\frac{2}{m}} + \cdots + \nu^{\frac{m}{m}})[1 + 2\nu + \cdots + n\nu^{n-1}] \\
&= \frac{1}{m}\frac{\nu^{\frac{1}{m}}(1 - \nu^{\frac{m}{m}})}{(1 - \nu^{\frac{1}{m}})}\nu^{-1}(\mathrm{I}a)_{\overline{n}|}
\end{aligned}
$$

$$= \frac{1}{m} \frac{(1-\nu)(\ddot{a}_{\overline{n}|} - n\nu^n)}{(\nu^{-\frac{1}{m}} - 1)(i\nu)}$$

Suppose every payment increases in arithmetic progression with a payment amount of $\frac{k}{m^2}$ made at time $t = k$.



The present value of these payments is:

$$(\mathrm{I}^{(m)}a)_{\overline{n}|}^{(m)} = \frac{\ddot{a}_{\overline{n}|}^{(m)} - n\nu^n}{i^{(m)}}. \qquad \text{Exercise 4-34}$$

Example: Suppose a deposit of \$1000 is made on the first of the months of January, February and March, \$1,200 at the beginning of each month in the second quarter, \$1,400 at the beginning of each month in the third quarter and \$1,600 each month in the fourth quarter. If the account has a nominal 8% rate of interest compounded quarterly, what is the balance at the end of the year?

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$$n = 4 \qquad m = 3 \qquad i = .08/4 = .02 \qquad \nu = (1.02)^{-1}$$

$$i^{(3)} = 3[(1.02)^{\frac{1}{3}} - 1] = .019868$$

The present value at $t = -1$ is

$$= 2400\left(\frac{1-(1.02)^{-4}}{.019868}\right) + 600\left(\frac{\frac{1-(1.02)^{-4}}{1-(1.02)^{-1}} - 4(1.02)^{-4}}{.019868}\right)$$

$$= 9,199.20 + 5,692.58 = \$14,891.78$$

At $t = 0$ the value is

$$(1.02)^{\frac{1}{3}}(14,891.78) = \$14,990.40$$

and its value at $t = 12$ is

$$(1.02)^{\frac{13}{3}}(14,891.78) = \$16,226.10$$

which is the balance at the end of the year.

Exercise 4-38:

A perpetuity provides payments every six-months starting today. The first payment is 1 and each payment is 3% greater than the immediately preceding payment. Find the present value of the perpetuity if the effective rate of interest is 8% per annum.

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Section 4.9 - Continuous Annuities with Varying Payments and/or Force of Interest

The accumulation function $a(t)$ and its reciprocal, the discount function $d(t) = \frac{1}{a(t)}$ play fundamental roles in the assessment of a series of payments when viewed from some specific time.

Consider a series of payments in which p_{t_j} denotes a payment made at time $t = t_j$ for $j = 1, 2, \cdots, n$. The present value of this series of payments is

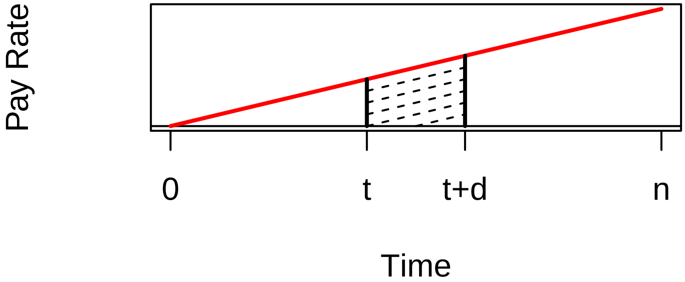
$$PV = \sum_{j=1}^{n} p_{t_j} \frac{1}{a(t_j)} = a(0)PV \qquad \text{because } a(0) = 1.$$

Likewise consider the future value of this sequence at the final payment, i.e. at $t = t_n$.

$$FV = \sum_{j=1}^{n} p_{t_j} \frac{a(t_n)}{a(t_j)} = a(t_n) \sum_{j=1}^{n} p_{t_j} \frac{1}{a(t_j)} = a(t_n)PV.$$

In general, when assessing the value of the sequence of payments from the perspective of some particular point in time $t = t_0$, the value is

This discussion is used to motivate a generalization of the continuous payment model discussed in section 4.5. In continuous payment settings the payments may not be smeared evenly over the number line (the assumption made in section 4.5). Some points in time may receive payments made at a higher rate than other points. This is described in terms of a function $f(t) \geq 0$ that expresses the smear of payment over the time line with $0 \leq t \leq n$. The smear could take on many shapes as illustrated below.


Pay Rate
0
t
t+d
n
Time

The present value of the smear of payment is

and the value at any other point in time $t = t_0$ is

$$Value(t_0) \quad = \quad \int_0^n f(t)\frac{a(t_0)}{a(t)}dt \quad = \quad a(t_0)PV.$$

Next attention is focused on the accumulation function $a(t)$. In continuous settings this is most often described in terms of the force of interest function δ_t.

Recall from section 1.9 that

$$\delta_t = \frac{a'(t)}{a(t)} \qquad \text{or equivalently} \qquad a(t) = e^{\int_0^t \delta_r dr}.$$

Therefore in terms of the force of interest function,

and when examining the payments from any other vision point $t = t_0$,

$$Value(t_0) \quad = \quad a(t_0)PV \quad = \quad e^{\int_0^{t_0} \delta_r dr} PV.$$

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Special Cases: (1) Constant Force of Interest

Here $\delta_t = \delta$ for all t. Moreover

$$a(t) = e^{\int_0^t \delta dr} = e^{\delta \int_0^t dr} = e^{\delta t}$$

which produces

(2) Force of Interest and Payment Function both Constant

Here $\delta_t = \delta$ and $f(t) = 1$ for all t . Therefore

$$PV = \int_0^n e^{-\delta t} dt = -\frac{1}{\delta} e^{-\delta t}\Big]_0^n =$$

a result found in section 4.5 where it was noted that n need not be an integer. Moreover if $n \to \infty$ then

$$PV = \overline{a}_{\overline{\infty}|} = \frac{1}{\delta}.$$

(3) Constant Force of Interest and a Linear Payment Function

Here $\delta_t = \delta$ and $f(t) = t$ for all t . Therefore using integration by parts

$$PV \equiv (\bar{\mathrm{I}}\ \bar{a})_{\overline{n|}} = \int_0^n t\ e^{-\delta t} dt = -\frac{t}{\delta} e^{-\delta t}\Big]_0^n + \frac{1}{\delta}\int_0^n e^{-\delta t} dt$$

$$= -\frac{n}{\delta} e^{-\delta n} + \left(-\frac{1}{\delta^2} e^{-\delta t}\Big]_0^n\right)$$

$$= \frac{1}{\delta^2}(1 - e^{-\delta n}) - \frac{n}{\delta} e^{-\delta n}$$

$$= \frac{1}{\delta}\Big(\frac{1 - \nu^n}{\delta} - n\nu^n\Big) \qquad \text{(recalling } \nu = e^{-\delta}\text{)}$$

Example: Continuous deposits are made of $100e^{t/5}$ for 5 years into an account with a constant force of interest of $\delta = .05$. What is the accumulated value at 5 years?

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First find the present value at $t = 0$:

$$PV = \int_0^5 \left(f(t) \frac{1}{a(t)} \right) dt$$

$$= 100 \int_0^5 e^{(.15)t} dt$$

$$= 100 \frac{1}{(.15)} e^{(.15)t} \Big]_0^5$$

$$= \left(\frac{100}{.15} \right) \left[e^{(.75)} - 1 \right]$$

Also,

$$a(5) = e^{\int_0^5 (.05) dr} = e^{(.25)}$$

Therefore

$$\begin{aligned} Value(5) &= a(5) PV \\ &= e^{(.25)} \left(\frac{100}{.15} \right) \left[e^{(.75)} - 1 \right] \\ &= 956.17094 \end{aligned}$$

So the balance is \$956.17 .

Exercise 4-43:

A one-year deferred continuously varying annuity is payable for 13 years. The rate of payment at time t is $t^2 - 1$ per annum, and the force of interest at time t is $(1 + t)^{-1}$. Find the present value of the annuity.

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