Chapter 03 - Basic Annuities

Section 3.0 - Sum of a Geometric Sequence

The form for the sum of a geometric sequence is:

$$\frac{\mathsf{Sum}(n)}{\mathsf{Sum}(n)} \equiv a + ar + ar^2 + ar^3 + \dots + ar^{n-1}$$

Here a = (the first term) n = (the number of terms) r = (the multiplicative factor between adjacent terms)

Note that

$$rSum(n) = ar + ar^{2} + \dots + ar^{n}$$
 and therefore
$$a + rSum(n) = a + ar + ar^{2} + \dots + ar^{n-1} + ar^{n}$$
$$= Sum(n) + ar^{n}.$$

Solving this equation for Sum(n) produces

$$(r-1)Sum(n) = a(r^n-1).$$

Therefore

$$Sum(n) = \begin{cases} \frac{a(1-r^n)}{(1-r)} = \frac{a(r^n-1)}{(r-1)} & \text{if } r \neq 1 \\ na & \text{if } r = 1 \end{cases}$$

We will refer to this formula with the abbreviation SGS. Example

$$100\nu + 100\nu^2 + 100\nu^3 + \dots + 100\nu^{30} = \frac{100\nu(1-\nu^{30})}{(1-\nu)}$$

So , for example, if $\nu = \frac{1}{1.1}$, then the above sum is

$$\frac{100(1.1)^{-1}(1-(1.1)^{-30})}{(1-(1.1)^{-1})} = 942.6914467$$

Section 3.1 - Annuity Terminology

Definition: An annuity is intervals of time.

Examples: Home Mortgage payments, car loan payments, pension payments.

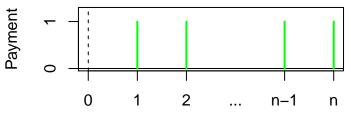
For an annuity - certain, the payments are made for a fixed (finite) period of time, called the term of the annuity. An example is monthly payments on a 30-year home mortgage.

For an contingent annuity, the payments are made until some event happens. An example is monthly pension payments which continue until the person dies.

The interval between payments (a month, a quarter, a year) is called the payment period. Section 3.2 - Annuity - Immediate (Ordinary Annuity)

In the annuity-Immediate setting

Generic Setting The amount of 1 is paid at the end of each of *n* payment periods.



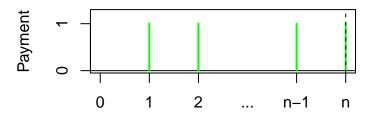
Time

The present value of this sequence of payments is

$$a_{\overline{n}|} \equiv a_{\overline{n}|i} \equiv \nu + \nu^2 + \nu^3 + \dots + \nu^n$$

$$= \frac{(1 - \nu^{n})}{i} \quad \text{because} \quad \frac{\nu}{1 - \nu} = \frac{(1 + i)^{-1}}{i(1 + i)^{-1}} = \frac{1}{i}$$

where *i* is the effective interest rate per payment period.



Viewing this stream of payments from the end of the last payment period, the accumulated value (future value) is

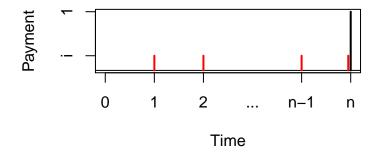
$$\begin{split} s_{\overline{n}|} &\equiv s_{\overline{n}|i} \equiv 1 + (1+i) + (1+i)^2 + \dots + (1+i)^{n-1} \\ &= \frac{1[(1+i)^n - 1]}{[(1+i) - 1]} \qquad \text{by SGS} \end{split}$$

Note also that $\nu = (1 + i)^{-1}$ implies

$$(1+i)^n a_{\overline{n}|} = (1+i)^n (\nu + \nu^2 + \dots + \nu^n)$$

= $(1+i)^{n-1} + (1+i)^{n-2} + \dots + 1$
= $s_{\overline{n}|}$

Invest 1 for *n* periods, paying *i* at the end of each period. If the principal is returned at the end, how does the present value of these payments relate to the initial investment?



$$1 = i\nu + i\nu^2 + \dots + i\nu^n + 1\nu'$$
$$= ia_{\overline{n}|} + \nu^n.$$

A relationship that will be used in a later chapter is

$$\frac{1}{|s_{\overline{n}|}|} + i = \frac{i}{(1+i)^n - 1} + i$$
$$= \frac{i+i(1+i)^n - i}{(1+i)^n - 1}$$
$$= \frac{i}{1 - \frac{1}{(1+i)^n}}$$
$$= \frac{i}{1 - \nu^n} = \frac{1}{|a_{\overline{n}|}|}$$

Example

Auto loan requires payments of \$300 per month for 3 years at a nominal annual rate of 9% compounded monthly. What is the present value of this loan and the accumulated value at its conclusion?

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There are n = 36 monthly payments and the effective monthly interest rate is .09/12 = .0075.

Present Value $= 300 a_{\overline{36}|.0075}$

= \$9,434.04

Accumulated Value =
$$300s_{\overline{36}|.0075}$$

= $300\frac{[(1+i)^n - 1]}{i} = 300\frac{[(1.0075)^{36} - 1]}{.0075}$
= \$12,345.81

Exercise 3-2:

The cash price of an automobile is \$10,000. The buyer is willing to finance the purchase at 18% convertible monthly and to make payments of \$250 at the end of each month for four years. Find the down payment that will be necessary.

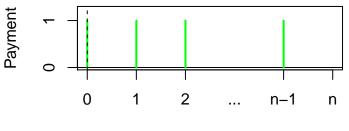
Exercise 3-6:

If $a_{\overline{n}|} = x$ and $a_{\overline{2n}|} = y$, express *i* as a function of *x* and *y*.

Section 3.3 - Annuity - Due

In the annuity-due setting

Generic Setting The amount of 1 is paid at the beginning of each of *n* payment periods.



The present value of this sequence of payments is

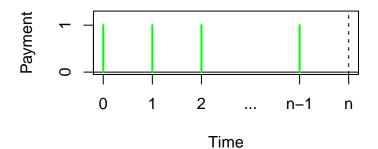
$$\ddot{\mathbf{a}}_{\overline{n}|} \equiv \ddot{\mathbf{a}}_{\overline{n}|i} \equiv \mathbf{1} + \nu + \nu^2 + \dots + \nu^{n-1}$$

$$=rac{(1-
u^n)}{d}$$

Note also that

$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$$

Now view these payments from the end of the last payment period.



The accumulated value at t = n is

$$\begin{split} \ddot{\mathbf{s}}_{\overline{n}|} &\equiv (1+i) + (1+i)^2 + \dots + (1+i)^n \\ &= \frac{(1+i)[(1+i)^n - 1]}{[(1+i) - 1]} \quad \text{(by SGS)} \quad = \frac{(1+i)^n - 1}{d}. \end{split}$$

Some additional useful relationships are:

$$\ddot{s}_{\overline{n}|} = (1+i)s_{\overline{n}|}$$
 and $rac{1}{\ddot{a}_{\overline{n}|}} = rac{1}{\ddot{s}_{\overline{n}|}} + d.$

Example

Starting on her 30^{th} birthday, a women invests x dollars every year on her birthday in an account that grows at an annual effective interest rate of 7%. What should x be if she wants this fund to grow to \$300,000 just before her 65^{th} birthday?

$$300 = x\ddot{s}_{\overline{35}|.07} = x\left(\frac{(1+.07)^{35}-1}{.07/(1+.07)}\right)$$
$$= x(147.91346) \quad \text{or} \quad x = 2.02821.$$
So payments should be \$2,028.21

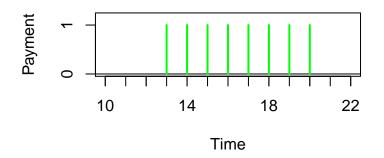
Exercise 3-8:

Find the present value of payments of \$200 every six months starting immediately and continuing through four years from the present, and \$100 every six months thereafter through ten years from the present, if $i^{(2)} = .06$.

Exercise 3-9:

A worker age 40 wishes to accumulate a fund for retirement by depositing \$3000 at the beginning of each year for 25 years. Starting at age 65 the worker plans to make 15 annual equal withdrawals at the beginning of each year. Assuming all payments are certain to be made, find the amount of each withdrawal starting at age 65 to the nearest dollar, if the effective interest rate is 8% during the first 25 years but only 7% thereafter.

Section 3.4 - Annuities Evaluated at Other Points in Time Example Setting:



The value of an annuity depends on the vision point (valuation point) of the analysis, i.e. the position of the valuation point relative to the payment stream. The above setting will be used for examples.

At the end of period 9 what is the value of these future payments? Here the answer is

$$u^3 a_{\overline{8}|} =
u^4 \ddot{a}_{\overline{8}|} =
u^{11} s_{\overline{8}|} =
u^{12} \ddot{s}_{\overline{n}|}$$

A deferred annuity is one that begins payments at some time in the future. Using the setting above, we could describe this stream of payments from the time t = 0 as

 $_{12}|a_{\overline{8}|} = (8 \text{ payment annuity immediate deferred 12 periods.})$

It could also be viewed as an annuity-due deferred 13 periods

$$_{13}|\ddot{a}_{\overline{8}|} =
u^{13}\ddot{a}_{\overline{8}|} = \ddot{a}_{\overline{21}|} - \ddot{a}_{\overline{13}|}$$

What is the accumulated value of this stream of payments at the end of period 24?

$$(1+i)^4 s_{\overline{8}|} = s_{\overline{12}|} - s_{\overline{4}|}$$
 (viewed as an annuity-immediate)
 $(1+i)^3 \ddot{s}_{\overline{8}|} = \ddot{s}_{\overline{11}|} - \ddot{s}_{\overline{3}|}$ (viewed as an annuity-due)

What is the value of this stream of payments at time t = 17?

$$u^3 s_{\overline{8}|} =
u^4 \ddot{s}_{\overline{8}|} = (1+i)^5 a_{\overline{8}|} = (1+i)^4 \ddot{a}_{\overline{8}|}$$

The value at t = 17 can also be expressed as, for example,

$$m{s}_{\overline{5}|}+m{a}_{\overline{3}|}=\ddot{m{s}}_{\overline{4}|}+\ddot{m{a}}_{\overline{4}|}$$

Exercise 3-14:

It is known that

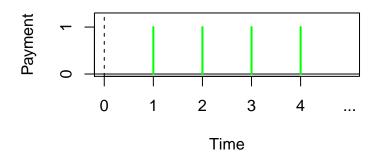
$$\frac{a_{\overline{7}|}}{a_{\overline{11}|}} = \frac{a_{\overline{3}|} + s_{\overline{\chi}|}}{a_{\overline{y}|} + s_{\overline{z}|}}.$$

Find x, y and z.

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Section 3.5 - Perpetuities

It is a perpetuity when the payments continue forever. Generic Example Perpetuity-Immediate:



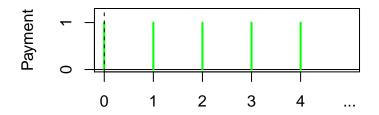
The present value of this infinite sequence of payments is $\nu + \nu^2 + \nu^3 + \nu^4 + \cdots$

The present value of the first *n* terms is, of course,

$$a_{\overline{n}|} = \frac{(1 - \nu^n)}{i}.$$

Since $0 < \nu < 1$, letting $n \rightarrow \infty$ produces

If the infinite stream of payments includes a payment made at the current time, then it becomes a perpetuity-due.



This generic perpetuity-due has a present value of

$$\ddot{a}_{\overline{\infty}|} = 1 + a_{\overline{\infty}|}$$

$$= 1 + \frac{1}{i} = \frac{(1+i)}{i}$$

The idea of a perpetuity is a useful computational tool. But there are very few financial instruments that have this exact structure. Note that

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u^n rac{1}{i} \ &=& rac{1-
u^n}{i}. \end{aligned}$$

Thus perpetuities can be useful in expressing the values of finite annuities.

Example: \$2000 is deposited at the beginning of each year for 15 years. Starting at the beginning of the 25th year, the account makes equal annual payments forever. If the account earns 6% effective per year what is the amount of these annual payments?

At t = 24 (beginning of the 25^{th} year) the account's value is

$$(2000)(1+.06)^9 \ddot{s}_{\overline{15}|.06} = (2000)(1.06)^9 \Big(\frac{(1+.06)^{15}-1}{.06/(1.06)} \Big) \\ = 83,367.43.$$

Set this equal to

This equation produces

$$x = \frac{(.06)(83, 367.43)}{(1.06)}$$
$$= \$4, 718.91.$$

Exercise 3-19:

A deferred perpetuity-due begins payments at time *n* with annual payments of \$1000 per year. If the present value of this perpetuity-due is equal to \$6561 and the effective rate of interest is i = 1/9, find *n*.

Section 3.6 - Unknown Time

The annuity-immediate present value formula, $a_{\overline{n}|}$, was developed assuming *n* is a positive integer. If a loan of *L* dollars is to be repaid with payments of *c* dollars per period, then

$$L = ca_{\overline{n}|} = c \frac{(1-\nu^n)}{i}$$
 or

represents the number of payments needed. But this *n* might not be an integer, i.e.

$$n_0 < n < n_0 + 1$$
.

In order to pay off the loan of *L*, one method is to make a drop payment (a partial payment) of *xc* where 0 < x < 1 at time $t = n_0 + 1$. This is done to complete the loan, producing

A second option for completing the loan is to make a balloon payment of c(1 + y) at time $t = n_0$ with 0 < y < 1. This will also complete the loan provided

$$L = c[a_{\overline{n_0}|} + y\nu^{n_0}]$$

In an analogous problem, suppose deposits of c are made at the beginning of each interest period with the goal of achieving a accumulated value of A at the end of the last period. How long would it take to achieve this goal?

Here

$$A = c\ddot{s}_{\overline{n}|} = c\Big(\frac{(1+i)^n - 1}{d}\Big)$$

produces

Again in this setting, this *n* may not be an integer, i.e. $n_0 < n < n_0 + 1$. Thus a balloon deposit at $t = n_0$ or a drop deposit at $t = n_0 + 1$ will be necessary to complete the accumulation goal.

Example: A couple sets up a college fund for their child in hopes of accumulating \$50,000 in the account by making annual deposits of \$2,000 in an account that earns an effective annual interest rate of 5%. If they make the first deposit today, how many deposits are needed to achieve their goal and what drop payment (deposit) is needed at the end?

= 16.07.

At t = 16 (one year after the 16^{th} deposit), the account contains

$$(2000)\ddot{s}_{\overline{16}|.05} = (2000) \Big(\frac{(1.05)^{16} - 1}{.05/(1.05)} \Big)$$

= \$49,680.73.

Thus a drop deposit of \$319.27 at t = 16 will achieve their goal at that point in time.

Exercise 3-24:

A loan of \$1000 is to be repaid by annual payments of \$100 to commence at the end of the fifth year and to continue thereafter for as long as necessary. Find the time and amount of the final payment, if the final payment is to be larger than the regular payments. Assume i = .045.

Section 3.7 - Unknown Rate of Interest

Suppose the annuity problem setting is one in which the interest rate is unknown, but the other characteristics are known. In these setting the equation to solve for *i* often takes the form:

$$a_{\overline{n}|}=g$$
 or $rac{1-\left(rac{1}{1+i}
ight)^n}{i}=g.$

where both g and n are known. This is clearly a difficult equation to solve for *i*.

The solution for i can be directly found using a financial calculator. That is the preferred mode of solution.

The textbook provides a simple but crude approximation

$$i \doteq rac{2(n-g)}{g(n+1)}.$$

If the setting produces an equation of the form

$$s_{\overline{n}|} = g^*$$

the solution is again obtained with a financial calculator. An approximation analogous to the one above is

Example: Twenty annual deposits of \$2000 will accumulate in an account to \$50,000 at the time of the last deposit. What is the annual effective interest rate of this account?

Using a financial calculator produces

i = .022854

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The approximation described above produces

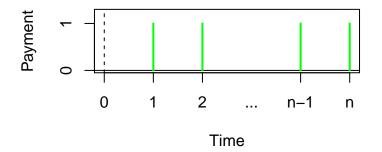
$$i \doteq \frac{2(25-20)}{25(19)} = .02105.$$

Exercise 3-29:

If $a_{\overline{2}|} = 1.75$, find an exact expression for *i*.

Section 3.8 - Varying Interest Rates

Often interest rates change from one period to the next. Let i_j denote the interest rate and $\nu_j = (1 + i_j)^{-1}$, the discount rate for the period which begins at t = j - 1 and ends at t = j.



The present value of payments of 1 at the end of each of n periods is

$$PV_n = \nu_1 + \nu_1\nu_2 + \nu_1\nu_2\nu_3 + \dots + \prod_{j=1}^n \nu_j$$

Now consider the accumulated value of these payments at time t = n:

$$FV_n = 1 + (1 + i_n) + (1 + i_n)(1 + i_{n-1}) + \dots + \prod_{j=1}^{n-1} (1 + i_{n+1-j})$$

$$= 1 + \sum_{t=1}^{n-1} \prod_{s=1}^{t} (1 + i_{n+1-s}).$$

If payments of 1 are made at the beginning of each period the present value is

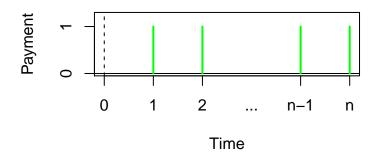
In this case, accumulated value of the deposits at t = n is

$$\ddot{FV}_n = (1 + i_n) + (1 + i_n)(1 + i_{n-1}) + \dots + \prod_{j=1}^n (1 + i_{n+1-j})$$

$$= \sum_{t=1}^{n} \prod_{s=1}^{t} (1 + i_{n+1-s}).$$

The scheme used so far in this section to attribute interest to each payment is called the portfolio method. With the portfolio method the interest rate i_j for the j^{th} interest period (stretching from t = j - 1 to t = j) is applied to each and every payment that is viewed as active during that j^{th} interest period. So when evaluating a stream of payments from a viewpoint of $t = t_0$, any payment that must pass through the j^{th} interest period to get to t_0 will have either ν_j or $(1 + i_j)$ applied to it as it passes through.

The portfolio method is contrasted with a second method which is called the yield curve method. When the yield curve method is applied, an interest rate of i_j is attached only to a payment made at time t = j and that rate (or its corresponding discount rate $\nu_j = (1 + i_j)^{-1}$) is applied to ALL periods in which that payment made at time t = j is active.



Using the generic annuity-immediate setting and the yield curve method consider the present value of these payments of 1:

$$PV_n = \nu_1 + (\nu_2)^2 + \dots + (\nu_n)^n =$$

Likewise,

$$\ddot{PV}_n = 1 + \nu_1 + (\nu_2)^2 + \dots + (\nu_{n-1})^{n-1} = 1 + PV_{n-1}.$$

Examining the accumulated values with the yield curve method produces

$$\ddot{FV}_n = (1 + i_{n-1}) + (1 + i_{n-2})^2 + \dots + (1 + i_0)^n$$
$$= \sum_{t=1}^n (1 + i_{n-t})^t,$$

where i_0 is the interest rate attached to a payment at t = 0. It also follows that

$$FV_n = 1 + \ddot{FV}_n - (1 + i_0)^n$$
.

Example:

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Use the yield curve method to find the accumulated value of twelve \$1000 payment at the last payment. Use 5% interest for the first 4 payments, 4% for the second 4 and 3% for the last 4 payments.

$$\begin{aligned} 000[(1+.05)^8 s_{\overline{4}|.05} + (1+.04)^4 s_{\overline{4}|.04} + s_{\overline{4}|.03}] \\ &= 1000 \Big[(1.05)^8 \Big(\frac{(1.05)^4 - 1}{.05} \Big) + (1.04)^4 \Big(\frac{(1.04)^4 - 1}{.04} \Big) \\ &+ \Big(\frac{(1.03)^4 - 1}{.03} \Big) \Big] \end{aligned}$$

= 1000[6.36802 + 4.96776 + 4.18363]

= \$15, 519.41.

Exercise 3-32(b):

Find the present value of an annuity-immediate which pays 1 at the end of each half-year for five years, if the payments for the first three years are subject to a nominal annual interest rate of 8% convertible semiannually and the payments for the last two years are subject to a nominal annual interest rate of 7% convertible semiannually.

Exercise 3-33:

Find the present value of an annuity-immediate which pays 1 at the end of each year for five years, if the interest rates are given by $i_t = .06 + .002(t - 1)$ for t = 1, 2, 3, 4, 5 where i_t is interpreted according to the (a) yield curve method (b) portfolio method