

Chapter 02 - Problems in Interest

Interest problems involve four quantities:

- (a) Principal Invested
- (b) Length of Investment
- (c) Rate of Interest (Discount)
- (d) Amount at the End of the Investment Period

Interest problems typically require solution for one or more of these variables, using values that are given or derived for the others.

Section 2.3 - Equations of Value

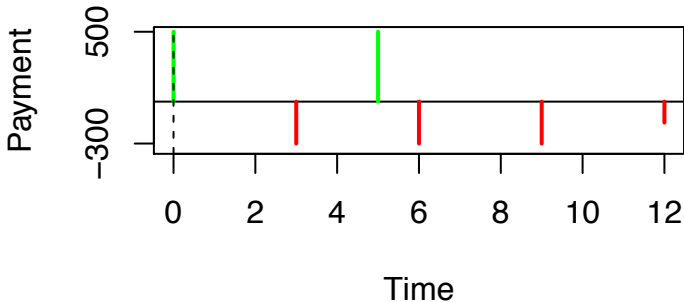
In these problem settings two equivalent investments are described in which one of the above quantities is unknown. Two valuation formulas are created (one for each investment) relative to some common comparison date. The two are equated to produce one equation in one unknown, an **equation of value**.

When using **compound interest**, the answer _____ on the _____
If **simple interest** is used,
the answer _____

A **Time Diagram** graphically displays each payment (money coming in and money going out) at its appropriate time point and projects each payment to one common time point (see examples to follow). It is a very useful device to set up an equation of value.

Example:

To receive a loan of \$500 today and another \$500 five years from today, you agree to pay \$300 at the end of years 3, 6, 9, and the remainder at the end of year 12. What is the final payment if interest is accrued at a nominal interest rate of 5% convertible semiannually?



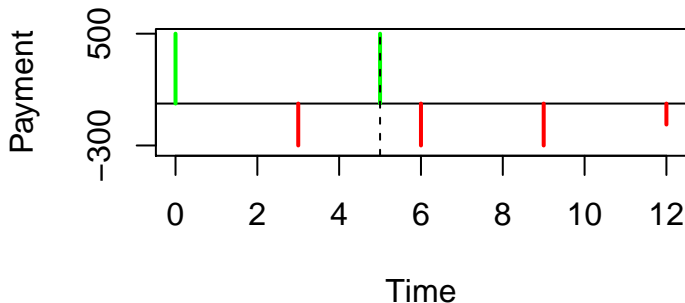
Using today ($t = 0$) as the point of comparison and $\nu = \frac{1}{(1 + \frac{.05}{2})}$, set the present value of both payment streams equal to one another. This yields

$$500 + 500\nu^{10} = 300\nu^6 + 300\nu^{12} + 300\nu^{18} + x\nu^{24}.$$

Then solving for x produces

$$x = \frac{500 + 500v^{10} - 300v^6 - 300v^{12} - 300v^{18}}{v^{24}}$$
$$= \$391.58.$$

Now repeat this problem solution using $t = 5$ as the comparison time point.



Now each payment is accumulated or discounted to $t = 5$. Again set the present values of the two payment streams equal to one another producing

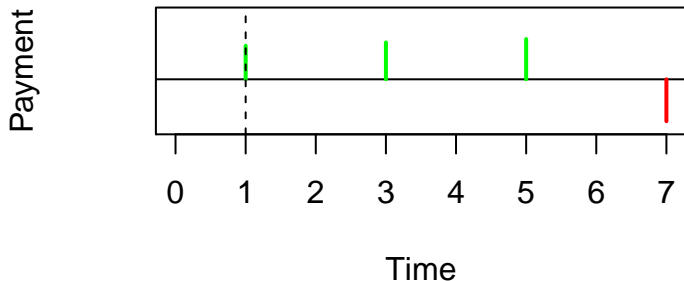
with $i = .025$. Then solving for x produces

$$x = \frac{500v^{-10} + 500 - 300v^{-4} - 300v^2 - 300v^8}{v^{14}}.$$

If both the numerator and denominator are multiplied by v^{10} , the expression is exactly the same expression as the solution on the previous page. So the answer is the same, namely \$391.58 .

Exercise 2.4:

An investor makes three deposits into a fund at the end of years 1, 3 and 5. The amount of the deposit at each time is $\$100(1.025)^t$. Find the balance of the account at the end of 7 years, if the nominal rate of discount convertible quarterly is $4/41$.



$$\nu = \left(1 - \frac{d^{(4)}}{4}\right) = \left(1 - \frac{1}{4}\right) = \frac{40}{41}$$

We use $t = 1$ as the vision point and form:

$$100(1.025) + 100(1.025)^3\nu^8 + 100(1.025)^5\nu^{16} = s\nu^{24}$$

$$s = \frac{100(1.025) + 100(1.025)^3\nu^8 + 100(1.025)^5\nu^{16}}{\nu^{24}}$$

$$= \$483.11$$

Exercise 2-2:

You have an inactive credit card with a \$1000 outstanding unpaid balance. This particular card charges interest at a rate of 18% compounded monthly. You are able to make a payment of \$200 one month from today and \$300 two months from today. Find the amount you will have to pay three months from today to completely pay off the credit card debt.

Exercise 2-33:

Person A signs a one-year note for \$1000 and receives \$920 from the bank. At the end of 6 months this person makes a payment of \$288. Assuming simple discount, to what amount does this reduce the face value of the note?

Section 2.4 - Unknown Time

In an account that has a nominal annual interest rate of $i^{(m)}$, compounded m times per year, a deposit of $A(0)$ will grow to

$$A(t) = A(0) \left(1 + \frac{i^{(m)}}{m}\right)^{mt}$$

by time t . So if A is a target amount for the account total, it will be achieved when

$$A = A(0) \left(1 + \frac{i^{(m)}}{m}\right)^{mt} \quad \text{or} \quad t = \frac{\ln(A) - \ln(A(0))}{m \ln\left(1 + \frac{i^{(m)}}{m}\right)}$$

The account will achieve the multiplication factor of k times the original deposit when

which does not depend on the original deposit amount, $A(0)$.

Another investment problem, in which we solve for time is when payments of s_j are made at times t_j , for $j = 1, 2, \dots, n$ and we set up an equation of value equating this stream to one payment of $(s_1 + s_2 + \dots + s_n)$ at time t . The objective being to find t . The equation of value is then

$$(s_1 + s_2 + \dots + s_n)\nu^t = s_1\nu^{t_1} + s_2\nu^{t_2} + \dots + s_n\nu^{t_n}.$$

Of course this can be solved for t using logarithms, provided the right side is not too complicated to compute,

$$t_0 = \frac{\ln(s_1\nu^{t_1} + s_2\nu^{t_2} + \dots + s_n\nu^{t_n}) - \ln(s_1 + s_2 + \dots + s_n)}{\ln(\nu)}$$

An approximate solution is found with the [Method of Equated Time](#) weighted mean

It can be shown that \bar{t} is always larger than a true solution, t_0 . (See the appendix of this chapter 's lecture notes.)

Example:

How long will it take for \$2000 to grow to \$3,200 in an account which has a nominal interest rate of 6% compounded quarterly?

$$4t \ln(1.015) = \ln\left(\frac{3200}{2000}\right) = .47$$

$$t = 7.89 \quad \text{years.}$$

Example:

The present value of a payment of \$100 at the end of n years plus another \$100 at the end of $2n$ years is \$90. If $i = .05$ is the annual effective interest rate, determine n .

$$\text{where } \nu = \frac{1}{1.05}.$$

Let $x = \nu^n$, then the above equation is The solution is then

$$x = \frac{-1 \pm \sqrt{1 + 4(.9)}}{2} = .57238 \quad (+ \text{ root needed}).$$

Thus

$$n = \frac{\ln(x)}{\ln(\nu)} = \frac{\ln(.57238)}{-\ln(1.05)} = 11.436 \quad \text{years.}$$

Exercise 2-9: A payment of n is made at the end of n years, $2n$ at the end of $2n$ years, \dots , n^2 at the end of n^2 years. Find the value of \bar{t} by the method of equated time.

Exercise 2-12: Fund A accumulates at a rate of 12% convertible monthly. Fund B accumulates with a force of interest of $\delta_t = t/6$. At time $t = 0$ equal deposits are made in each fund. Find the next time the two funds have equal balances.

Section 2.5 - Unknown Rate of Interest

Suppose the principal invested, $A(0)$, accumulates to $A(t)$ at the end of t years. If the interest is compounded m times per year, the solution for the **nominal annual interest rate** is found via

$$A(0) \left(1 + \frac{i^{(m)}}{m} \right)^{mt} = A(t),$$

which produces

The solution for the **annual effective interest rate** is

$$i = \left(\frac{A(t)}{A(0)} \right)^{\frac{1}{t}} - 1.$$

Example:

Suppose \$1000 is invested for 2.5 years and grows in that time to \$1220 in an account that is convertible semiannually. What are the effective and nominal interest rates of this investment?

Effective:

$$i = \left(\frac{1220}{1000} \right)^{\frac{1}{2.5}} - 1 = .08278.$$

Nominal:

$$i^{(2)} = 2 \left[\left(\frac{1220}{1000} \right)^{\frac{1}{5}} - 1 \right] = .08114.$$

Another type of problem involves setting up an equation of value which is a polynomial equation in an unknown interest or discount rate.

Example: \$200 at the end of 2 years, \$400 at the end of 4 years and \$600 at the end of 6 years accumulates to \$1,400 at the end of 6 years. What is the annual effective rate of interest?

Letting $x = (1 + i)^2$, the equation becomes

The solution is thus

$$x = \frac{-2 \pm \sqrt{4 + (4)(4)}}{2} = \sqrt{5} - 1.$$

So

$$i = \sqrt{(\sqrt{5} - 1)} - 1 = .1118.$$

Exercise2-18: The sum of the accumulated value of 1 at the end of three years at a certain effective interest rate of i and the present value of 1 to be paid at the end of three years at an effective rate of discount that is numerically equal to i is 2.0096. Find i .

Exercise 2-22: A bill for \$100 is purchased for \$96 three months before it is due. Find (a) the nominal rate of discount convertible quarterly earned by the purchaser and (b) the annual effective rate of interest earned by the purchaser.

Section 2.6 - Determination of Time Periods

The time period of an investment is determined by counting the number of days that it is in force. There are three methods commonly used to make these counts.

(1) Exact Simple Interest (actual/actual)

The exact number of days in force is compared to a 365 day year.

(2) Ordinary Simple Interest (30/360)

is compared to a 360 day year, where Y_j , M_j and D_j are the year, month and day when the investment is made ($j = 1$) and withdrawn ($j = 2$).

(3) Banker's Rule (actual/360)

The exact number of days in force is compared to a 360 day year.

Example:

\$1000 is to be invested from March 15 to September 4 of 2018. Find the simple interest earned at 6% by all three methods.

(1) Exact Count: produces $247 - 74 = 173$ days

$$1000(.06)\left(\frac{173}{365}\right) = 28.44$$

(2) Ordinary Simple:

$$1000(.06)\left(\frac{169}{360}\right) = 28.17$$

(3) Banker's Rule:

$$1000(.06)\left(\frac{173}{360}\right) = 28.83$$

Miscellaneous Problems

Exercise 2-29

A manufacturer sells a product to a retailer who has the option of paying 30% below retail price immediately or 25% below the retail price in 6 months. Find the annual effective interest rate at which the retailer is indifferent between these two options.

The retailer will be indifferent if

which yields

$$(1 + i)^{\frac{1}{2}} = 1.0714286$$

or

$$i = .14796$$

Exercise 2-30: You deposit \$1000 into a bank account which credits interest at a nominal annual rate of i convertible semiannually for the first 7 years and a nominal annual rate of $2i$ convertible quarterly for all years thereafter. The accumulated amount in the account at the end of 5 years is x . The accumulated amount at the end of 10.5 years is \$1980. Calculate x to the nearest dollar.

Exercise 2-31: Fund A accumulates at 6% effective and fund B accumulates at 8% effective. At the end of 20 years the total in the two funds is \$2000. At the end of 10 years the amount in fund A is half that in fund B. What is the total of the two funds at the end of 5 years?

Appendix to Chapter 2

Theorem: Let y_1, y_2, \dots, y_n denote a set of positive values and w_1, w_2, \dots, w_n denote a set of positive weights. The weighted arithmetic mean is larger than the weighted geometric mean, i.e.

$$\frac{\sum_{i=1}^n w_i y_i}{\sum_{i=1}^n w_i} > \left[\prod_{i=1}^n y_i^{w_i} \right]^{\frac{1}{\sum_{i=1}^n w_i}}$$

as long as at least two of the y_i 's differ.

Proof: Without loss of generality assume $\sum_{i=1}^n w_i = 1$. We seek to show

$$\sum_{i=1}^n w_i y_i > \prod_{i=1}^n y_i^{w_i} \quad (1_A)$$

Consider the case $n=2$ with $y_1 \neq y_2$. Then (1_A) becomes

$$w_1 y_1 + (1 - w_1) y_2 > y_1^{w_1} y_2^{1-w_1} \quad \text{or}$$

$$w_1(t) + (1 - w_1) > t^{w_1}$$

where $t = y_1/y_2$ is greater than zero but not equal to 1. Let

$$g(t) = w_1 t + 1 - w_1 - t^{w_1} \quad \text{and note that}$$

$$g'(t) = w_1(1 - t^{w_1-1}) \quad \text{and}$$

$$g''(t) = -w_1(w_1 - 1)t^{w_1-2}.$$

Since the weights are positive and sum to 1,

$$0 < w_1 < 1 \quad \text{and} \quad w_1 - 2 < w_1 - 1 < 0.$$

Thus $g(t)$ is a convex function because $g''(t) > 0$ for all $t > 0$. Also note that

$$g'(t) \begin{cases} < 0 & \text{for } 0 < t < 1 \\ = 0 & \text{for } t = 1 \\ > 0 & \text{for } t > 1. \end{cases}$$

So $g(t)$ is strictly decreasing for $0 < t < 1$ and strictly increasing for $t > 1$. At $t = 1$, $g(1) = 0$. So $g(t) > 0$ for all $t > 0$ except when $y_1 = y_2$, that is, $t = 1$. This proves (1_A) holds for $n = 2$.

The case for general n is proved recursively. Assume (1_A) holds for $n - 1$, that is

$$\sum_{i=1}^{n-1} w_i y_i > \prod_{i=1}^{n-1} y_i^{w_i} \quad \text{with} \quad \sum_{i=1}^{n-1} w_i = 1 \quad (2_A)$$

Then when $\sum_{i=1}^n w_i = 1$,

$$\begin{aligned} \sum_{i=1}^n w_i y_i &= \left[\sum_{i=1}^{n-1} \frac{w_i y_i}{(1 - w_n)} \right] (1 - w_n) + w_n y_n \\ &> \left[\prod_{i=1}^{n-1} y_i^{\frac{w_i}{1 - w_n}} \right] (1 - w_n) + w_n y_n \quad \text{by } (2_A) \end{aligned}$$

$$> \left[\prod_{i=1}^{n-1} y_i^{\frac{w_i}{1-w_n}} \right] (1-w_n) y_n^{w_n} \quad \text{by the } n = 2 \text{ case}$$

$$> \prod_{i=1}^n y_i^{w_i}$$

This completes the proof for general n .

Special Case - All Weights Equal: Letting $w_i = 1/n$, the theorem produces

$$\bar{y} > \left(\prod_{i=1}^n y_i \right)^{1/n},$$

that is, the arithmetic mean is larger than the geometric mean when the values are all positive and at least two are different.

Special Case with $w_i = s_i$ and $y_i = \nu^{t_i}$: The theorem shows that

$$\begin{aligned}\frac{\sum_{i=1}^n s_i \nu^{t_i}}{\sum_{i=1}^n s_i} &> \left[\prod_{i=1}^n (\nu^{t_i})^{s_i} \right]^{\frac{1}{\sum_{i=1}^n s_i}} \\ &= \nu^{\frac{\sum_{i=1}^n t_i s_i}{\sum_{i=1}^n s_i}} \\ &= \nu^{\bar{t}}\end{aligned}\quad (3_A)$$

By its definition, the actual solution, t_0 , satisfies

$$(s_1 + s_2 + \cdots + s_n) \nu^{t_0} = \sum_{i=1}^n s_i \nu^{t_i}.$$

But (3_A) shows that

$$\sum_{i=1}^n s_i \nu^{t_i} > (s_1 + s_2 + \cdots + s_n) \nu^{\bar{t}}$$

It follows that the approximation, $\bar{t} > t_0$, the actual solution because $0 < \nu < 1$.

Exercise: Two funds A and B start with the same amount. Fund A grows at an annual interest rate of $i > 0$ for n years and at an annual rate of $j > 0$ for the next n years. Fund B grows at an annual rate of k for $2n$ years. Fund A equals 1.5 times fund B after n years. The amounts in the two funds are equal after $2n$ years. Which of the following are true?

- (a) $j < k < i$ (b) $k < \frac{i+j}{2}$ (c) $j = k \left(\frac{2}{3}\right)^{\frac{1}{n}}$
