On Bootstrap Inconsistency and Its Repair via the Biased Bootstrap

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Outline

Weak Convergence of Random Distributions

Some Examples of Bootstrap Inconsistency

Repairing Inconsistency via the Biased Bootstrap

Examples Revisited

Simulations.

Some Old Friends . . .

Theorem (Slutsky)

$$\left.\begin{array}{l} X_n \xrightarrow{d} X \\ Y_n \xrightarrow{\text{Pr}} c \end{array}\right\} \implies (X_n, Y_n) \xrightarrow{d} (X, c)$$

Theorem (H. Rubin)

$$\left.\begin{array}{l} X_n \xrightarrow{d} X \\ \\ g_n(x_n) \to g(x) \quad \textit{for all}^* \ x_n \to x \end{array}\right\} \implies g_n(X_n) \xrightarrow{d} g(X)$$

* *For all* $x \in A$ *and all sequences* $x_n \to x$, *where* $\Pr(X \in A) = 1$.

. . . Applied to Random Distributions . . .

Setup:

$S_1 \quad S_2 \quad S_3$ separable metric spaces

$\mathscr{P}_1 \quad \mathscr{P}_2 \quad \mathscr{P}_3$ spaces of probability measures on their Borel sets

$\rho_i =$ metric on $\mathscr{P}_i$ metrizing weak convergence

$X_n =$ random elements of S_1

$\mathbb{Q}_n =$ random elements of $\mathscr{P}_2$

$\psi_n =$ measurable mappings of $(S_1 \times \mathscr{P}_2)$ into $\mathscr{P}_3$

Assume:

$$X_n \xrightarrow{\text{d}} X$$

$$\mathbb{Q}_n \xrightarrow{\text{Pr}} Q \text{ (nonrandom)}$$

$$\psi_n(x_n, Q_n) \to \psi(x, Q) \quad \text{for all}^* \ (x_n, Q_n) \to (x, Q)$$

... Applied to Random Distributions ...

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...

$$\left.\begin{array}{r} X_n \xrightarrow{\text{d}} X \\ \mathbb{Q}_n \xrightarrow{\text{Pr}} Q \end{array}\right\} \xRightarrow{\text{Slutsky}} (X_n, \mathbb{Q}_n) \xrightarrow{\text{d}} (X, Q)$$

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$$\text{for all}^* \ (x_n, Q_n) \to (x, Q)$$

$$\xRightarrow{\text{H.Rubin}} \psi_n(X_n, \mathbb{Q}_n) \xrightarrow{\text{d}} \psi(X, Q)$$

$S_1 \quad S_2 \quad S_3$ separable metric spaces

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Proposition

Assume:

$$X_n \xrightarrow{d} X$$

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$$\psi_n(x_n, Q_n) \to \psi(x, Q) \quad \textit{for all}^* \ (x_n, Q_n) \to (x, Q)$$

Then:

$$\psi_n(X_n, \mathbb{Q}_n) \xrightarrow{d} \psi(X, Q)$$

X_n, Y_n	random elements of S_1 and S_2, resp.
$g_n(x, y)$	S_3-valued function on $S_1 \times S_2$
$\mathscr{A}_n$	sub-σ-algebra of underlying probability space

Corollary

Assume:

$$X_n \in \mathscr{A}_n \quad \textit{and} \quad X_n \xrightarrow{d} X \sim P$$

$$\mathscr{L}(Y_n | \mathscr{A}_n) \xrightarrow{\Pr} Q \text{ (nonrandom)}$$

$$g_n(x_n, y_n) \to g(x, y) \quad \textit{for all}^* \ (x_n, y_n) \to (x, y)$$

Then:

$$\mathscr{L}\big(g_n(X_n, Y_n)\big|\mathscr{A}_n\big) \xrightarrow{d} \mathscr{L}\big(g(X, Y)\big|X\big), \quad X \sim P, \quad Y \sim Q, \quad X \perp\!\!\!\perp Y.$$

▸ Proof

Examples: Generalities

Each of our examples follows the same pattern

- $X_{1:n} := X_1, \ldots, X_n \sim$ i.i.d.
- $X_1^*, \ldots, X_n^*$ a with-replacement random sample from $X_{1:n}$
- $S_n \in \sigma(X_{1:n})$ and $S_n \xrightarrow{\text{d}} Z \sim N(0, \Sigma)$
- $\mathscr{L}(S_n^* | X_{1:n}) \xrightarrow{\text{Pr}} N(0, \Sigma)$
- $T_n = g_n(0, S_n; \theta)$
- $T_n^* = g_n(S_n, S_n^*; \theta)$
- $g_n(x_n, y_n; \theta) \to g(x, y; \theta)$ for all* $(x_n, y_n) \to (x, y)$

Thus

- $T_n \xrightarrow{\text{d}} g(0, Z; \theta) \quad Z \sim N(0, \Sigma)$
- $\mathscr{L}(T_n^* | X_{1:n}) \xrightarrow{\text{d}} \mathscr{L}(g(Z, W; \theta) | Z) \quad Z, W \sim$ i.i.d. $N(0, \Sigma)$.

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For most values of θ:

$$g(x, y; \theta) = g(0, y; \theta)$$

so that

$$\begin{aligned}\mathscr{L}\big(g(Z, W; \theta) \big| Z\big) &= \mathscr{L}\big(g(0, W; \theta) \big| Z\big) \\ &= \mathscr{L}\big(g(0, W; \theta)\big) = \mathscr{L}\big(g(0, Z; \theta)\big) ✔\end{aligned}$$

For other values of θ:

$g(x, y; \theta)$ depends on x and

$$\mathscr{L}\big(g(Z, W; \theta) \big| Z\big) \text{ is random (varies with } Z) ✘$$

Example: Hodges' Estimator

$X_1, \ldots, X_n \sim$ i.i.d. $(\mu, 1)$ $\qquad 0 \le b < 1$

$$\hat{\mu}_n = \begin{cases} \bar{X}_n & \text{if } |\bar{X}_n| > n^{-1/4} \\ b\,\bar{X}_n & \text{if } |\bar{X}_n| \le n^{-1/4} \end{cases}$$

$$S_n = n^{1/2}(\bar{X}_n - \mu) \qquad S_n^* = n^{1/2}(\bar{X}_n^* - \bar{X}_n)$$

$$S_n \xrightarrow{\text{d}} Z \sim N(0,1) \qquad \mathscr{L}(S_n^* | X_{1:n}) \xrightarrow{\text{Pr}} N(0,1)$$

$$T_n = n^{1/2}(\hat{\mu}_n - \mu) \qquad T_n^* = n^{1/2}(\hat{\mu}_n^* - \bar{X}_n)$$

$$g_n(x, y; \mu) = y - (1-b)(x + y + n^{1/2}\mu)\,\mathrm{I}(|x + y + n^{1/2}\mu| \le n^{1/4})$$

$$g(x, y; \mu) = \begin{cases} y, & \text{if } \mu \neq 0, \\ by + (1-b)x, & \text{if } \mu = 0 \end{cases}$$

Example: Hodges' Estimator (p2)

$$g(x, y; \mu) = \begin{cases} y, & \text{if } \mu \neq 0, \\ by + (1-b)x, & \text{if } \mu = 0 \end{cases}$$

$$T_n \xrightarrow{\text{d}} g(0, Z; \mu) = \begin{cases} Z & \text{if } \mu \neq 0 \\ bZ & \text{if } \mu = 0 \end{cases} \qquad Z \sim N(0,1)$$

$$\mathscr{L}(T_n^* | X_{1:n}) \xrightarrow{\text{d}} \mathscr{L}\big(g(Z, W; \mu) \big| Z\big) \quad Z, W \sim \text{i.i.d. } N(0,1)$$

$\mu \neq 0 :\ g(x, y; \mu) = g(0, y; \mu) = y$

$$\mathscr{L}\big(g(Z, W; \mu) \big| Z\big) = \mathscr{L}(W|Z) = \mathscr{L}(W) = N(0,1) \ ✔$$

$\mu = 0 :\ g(x, y; \mu) = by + (1-b)x$

$$\mathscr{L}\big(g(Z, W; 0) \big| Z\big) = \mathscr{L}\big(bW + (1-b)Z \big| Z\big) = N\big((1-b)Z, b^2\big) \ ✘$$

Example: Lindley's Estimator

$$X_1, \ldots, X_n \sim \text{i.i.d. } (\mu, I_p) \quad (p \geq 4)$$

$$\hat{\mu}_n = m(\bar{X}_n)1_p + \left\{1 - \frac{p-3}{n\|\bar{X}_n - m(\bar{X}_n)1_p\|^2}\right\}\{\bar{X}_n - m(\bar{X}_n)1_p\}$$

where

$$m(x) := \frac{1}{p}\sum_{i=1}^{p} x^{(i)}, \quad x \in \mathbb{R}^p$$

Example: Lindley's Estimator (p2)

$$S_n = n^{1/2}(\bar{X}_n - \mu) \qquad S_n^* = n^{1/2}(\bar{X}_n^* - \bar{X}_n)$$

$$S_n \xrightarrow{\text{d}} Z \sim N(0, I_p) \qquad \mathscr{L}(S_n^* | X_{1:n}) \xrightarrow{\text{Pr}} N(0, I_p)$$

$$T_n = n^{1/2}(\hat{\mu}_n - \mu) \qquad T_n^* = n^{1/2}(\hat{\mu}_n^* - \bar{X}_n)$$

$$g_n(x, y; \mu) = y - \frac{(p-3)[(x+y) - m(x+y)1_p + n^{1/2}\{\mu - m(\mu)1_p\}]}{\|(x+y) - m(x+y)1_p + n^{1/2}\{\mu - m(\mu)1_p\}\|^2}$$

$$g(x, y; \mu) = \begin{cases} y & \text{if } \mu \neq m(\mu)1_p \\ y - \dfrac{(p-3)[(x+y) - m(x+y)1_p]}{\|(x+y) - m(x+y)1_p\|^2} & \text{if } \mu = m(\mu)1_p \end{cases}$$

Example: Lindley's Estimator (p3)

$$T_n \xrightarrow{\text{d}} g(0, Z; \mu) \quad Z \sim N(0, I_p)$$

$$\mathscr{L}(T_n^* | X_{1:n}) \xrightarrow{\text{d}} \mathscr{L}\big(g(Z, W; \mu)\big| Z\big) \quad Z, W \sim \text{i.i.d. } N(0, I_p)$$

$\mu \neq m(\mu)1_p$: $g(x, y; \mu) = g(0, y; \mu) = y$

$$T_n \xrightarrow{\text{d}} Z \sim N(0, I_p)$$

$$\mathscr{L}\big(g(Z, W; \mu)\big| Z\big) = \mathscr{L}(W|Z) = \mathscr{L}(W) = N(0, I_p) \checkmark$$

$\mu = m(\mu)1_p$:

$$T_n \xrightarrow{\text{d}} Z - \frac{(p-3)\{Z - m(Z)1_p\}}{\|Z - m(Z)1_p\|^2}, \quad Z \sim N(0, I_p)$$

$$\mathscr{L}\big(g(Z, W; \mu)\big| Z\big) = \mathscr{L}\left(W - \frac{(p-3)\big[(Z+W) - m(Z+W)1_p\big]}{\|(Z+W) - m(Z+W)1_p\|^2} \middle| Z\right) \times$$

Example: The LASSO (Correlation Model)

$(X_1, Y_1), \ldots, (X_n, Y_n)$ i.i.d. $\qquad E\left[(Y_i - X_i^T\beta)X_i\right] = 0$

$$\hat{\beta}_n = \underset{b}{\operatorname{argmin}} \left\{ \sum_{i=1}^{n} (Y_i - X_i^T b)^2 + \lambda_n \sum_{j=1}^{p} \left| b^{(j)} \right| \right\}$$

Assume: $E\left(\|(X_i, Y_i)\|^4\right) < \infty$

$V := \operatorname{Var}\left[(Y_i - X_i^T\beta)X_i\right]$ positive definite

$C := E\left(X_i X_i^T\right)$ positive definite

$n^{-1/2}\lambda_n \to \lambda_0$

Let: $\Sigma = C^{-1} V C^{-1}$

Bootstrapping Cases

$(X_1^*, Y_1^*), \ldots, (X_n^*, Y_n^*)$ is a with-replacement random sample from $(X, Y)_{1:n} = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}$.

Example: LASSO (Correlation Model) (p2)

$$C_n = n^{-1} \sum_{i=1}^{n} X_i X_i^T \qquad C_n^* = n^{-1} \sum_{i=1}^{n} X_i^* X_i^{*T}$$

$$\bar{\beta}_n = C_n^{-1} n^{-1} \sum_{i=1}^{n} X_i Y_i \quad (LSE) \qquad \bar{\beta}_n^* = C_n^{*-1} n^{-1} \sum_{i=1}^{n} X_i^* Y_i^*$$

$$S_n = n^{1/2}(\bar{\beta}_n - \beta) \qquad S_n^* = n^{1/2}(\bar{\beta}_n^* - \bar{\beta}_n)$$

$$T_n = n^{1/2}(\hat{\beta}_n - \beta) \qquad T_n^* = n^{1/2}(\hat{\beta}_n^* - \bar{\beta}_n)$$

$$C_n \xrightarrow{\text{Pr}} C \qquad C_n^* \xrightarrow{\text{Pr}} C$$

$$S_n \xrightarrow{\text{d}} Z \sim N(0, \Sigma) \qquad \mathscr{L}(S_n^* | X_{1:n}) \xrightarrow{\text{Pr}} N(0, \Sigma)$$

Example: LASSO (Correlation Model) (p3)

$$g_n(x, y, C; \beta) =$$

$$y + \underset{u}{\operatorname{argmin}} \Big\{ u^T C u + n^{-1/2} \lambda_n \sum_{j=1}^{p} \Big(|n^{1/2} \beta^{(j)} + (x + y + u)^{(j)}| - |n^{1/2} \beta^{(j)}| \Big) \Big\}$$

$$g(x, y, C; \beta) =$$

$$y + \underset{u}{\operatorname{argmin}} \Big\{ u^T C u + \lambda_0 \sum_{j=1}^{p} \Big(u^{(j)} \operatorname{sign}(\beta^{(j)}) + \big|(x + y + u)^{(j)}\big| \, \mathrm{I}\big(\beta^{(j)} = 0\big) \Big) \Big\}$$

Note: $\beta^{(j)} \neq 0$ for all $j = 1, \ldots, p \implies g(x, y, C; \beta) = g(0, y, C; \beta)$

Example: LASSO (Correlation Model) (p4)

$$T_n \xrightarrow{d} g(0, Z, C; \beta), \quad Z \sim N(0, \Sigma)$$

$$\mathscr{L}(T_n^* | X_{1:n}) \xrightarrow{d} \mathscr{L}\big(g(Z, W, C; \beta) \big| Z\big) \quad Z, W \sim \text{i.i.d. } N(0, \Sigma)$$

$\beta^{(j)} \neq 0$ for all $j = 1, \ldots, p$: $g(x, y, C; \beta) = g(0, y, C; \beta)$

$$\mathscr{L}\big(g(Z, W, C; \beta) \big| Z\big) = \mathscr{L}\big(g(0, W, C; \beta) \big| Z\big) = \mathscr{L}\big(g(0, W, C; \beta)\big) \checkmark$$

$\beta^{(j)} = 0$ for at least one $j = 1, \ldots, p$:

$\mathscr{L}\big(g(Z, W, C; \beta) \big| Z\big)$ varies with Z and is random ✘

A Way to Repair the Bootstrap

Why is the bootstrap inconsistent in these examples?

- Limiting distribution of T_n is discontinuous in μ.
- Resampling from empirical distribution with mean $\bar{X}_n \neq \mu$.

Idea

- Resample from a distribution with mean $\tilde{\mu}_n$ chosen to fix bootstrap inconsistency.

Candidate Family of Distributions

- Weighted empirical distribution with weight w_i on X_i, $i = 1, \dots, n$.
- For a given value of μ, choose the weights to
  - minimize Kullback-Leibler divergence from empirical distribution
  - subject to constraint $\sum_{i=1}^{n} w_i X_i = \mu$.
- Defines a pseudo-parametric family of distributions $\{\hat{P}_\mu\}$.

Theorem

Suppose:

- $\Theta = \uplus_{i \in \mathcal{I}} \Theta_i \subset \mathbb{R}^p$.
- $\mathscr{L}(S_n^* | X_{1:n}) \xrightarrow{P_\theta^n} Q_\theta$ *for all* $\theta \in \Theta$.
- $Q_\theta(A_\theta) = 1$ *for all* $\theta \in \Theta$.
- $\tilde{\theta}_n \xrightarrow{P_\theta^n} \theta$ *and* $P_\theta^n(\tilde{\theta}_n \in \Theta_i) \to 1$ *for all* $\theta \in \Theta_i$ *and* $i \in \mathcal{I}$.
- *For all* $i \in \mathcal{I}$ *and all* $\theta \in \Theta_i$, $g_n(x_n; \theta_n) \to g(x; \theta)$ *for all sequences* $x_n \to x \in A_\theta$ *and* $\theta_n \to \theta$ *with* $\{\theta_n, n \geq 1\} \subset \Theta_i$.

Then for all $\theta \in \Theta$: $\mathscr{L}\big(g_n(S_n^*, \tilde{\theta}_n) \big| X_{1:n}\big) \xrightarrow{P_\theta^n} \mathscr{L}\big(g(Z, \theta)\big)$, $Z \sim Q_\theta$.

Corollary

If $\mathscr{L}(S_n) \xrightarrow{P_\theta^n} Q_\theta$ *for all* $\theta \in \Theta$, *then the conditional distribution of* $T_n^* = g_n(S_n^*, \tilde{\theta}_n)$ *given* $X_{1:n}$ *consistently estimates the distribution of* $T_n = g_n(S_n, \theta)$ *under all values of* $\theta \in \Theta$.

Theorem

Suppose:

- $\Theta = \uplus_{i \in \mathcal{I}} \Theta_i \subset \mathbb{R}^p$.
- $\mathscr{L}(S_n^* | X_{1:n}) \xrightarrow{P_\theta^n} Q_\theta$ for all $\theta \in \Theta$.
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Assume

$$X_1, \dots, X_n \sim \textit{i.i.d.}\ (\mu, \Sigma)\ \textit{(finite)}$$

$$\tilde{\mu}_n \textit{ is } \sqrt{n}\textit{-consistent for } \mu$$

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$$X_1^*, \dots, X_n^* \sim \textit{i.i.d.}\ \hat{P}_{\tilde{\mu}_n} \quad \textit{conditional on } X_{1:n} = \{X_1, \dots, X_n\}$$

Then

$$\mathscr{L}\left(n^{1/2}(\bar{X}_n^* - \tilde{\mu}_n) \middle| X_{1:n}\right) \xrightarrow{\text{Pr}} N(0, \Sigma)$$

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Examples Revisited

Example (Hodges Estimator)

- $\Theta_0 = \{0\}$, $\Theta_1 = (-\infty, 0) \cup (0, \infty)$.
- $g_n(x;\mu) = x - (1-b)(x + n^{1/2}\mu)\,\mathrm{I}(|x + n^{1/2}\mu| \le n^{1/4})$
- $g(x;\mu) = x - (1-b)x\,\mathrm{I}(\mu = 0) = \begin{cases} x & \text{if } \mu \neq 0 \\ bx & \text{if } \mu = 0 \end{cases}$
- $\tilde{\mu}_n = \bar{X}_n I(|\bar{X}_n| \ge a_n)$ with $a_n \to 0$ and $n^{1/2} a_n \to \infty$.

Examples Revisited

Example (Lindley's Estimator)

- $\Theta_0 = \{\mu \in \mathbb{R}^p : \mu = m(\mu)1_p\}$, $\Theta_1 = \{\mu \in \mathbb{R}^p : \mu \neq m(\mu)1_p\}$.
- $$g_n(x;\mu) = x - \frac{(p-3)\left[(x) - m(x)1_p + n^{1/2}\{\mu - m(\mu)1_p\}\right]}{\left\|(x) - m(x)1_p + n^{1/2}\{\mu - m(\mu)1_p\}\right\|^2}$$
- $$g(x;\mu) = \begin{cases} x, & \text{if } \mu \neq m(\mu)1_p, \\ x - \dfrac{(p-3)[(x) - m(x)1_p]}{\|(x) - m(x)1_p\|^2}, & \text{if } \mu = m(\mu)1_p. \end{cases}$$
- $$\tilde{\mu}_n = \begin{cases} \bar{X}_n & \text{if } \|\bar{X}_n - m(\bar{X}_n)1_p\| > a_n \\ m(\bar{X}_n)1_p & \text{if } \|\bar{X}_n - m(\bar{X}_n)1_p\| \leq a_n \end{cases}$$

  with $a_n \to 0$ and $n^{1/2}a_n \to \infty$.

Examples Revisited

Example (LASSO)

- $\Theta = \mathbb{R}^p = \biguplus_{J \in \mathcal{I}} \Theta_J$ with $\mathcal{I} = \big\{J : J \subset \{1, \dots, p\}\big\}$ and

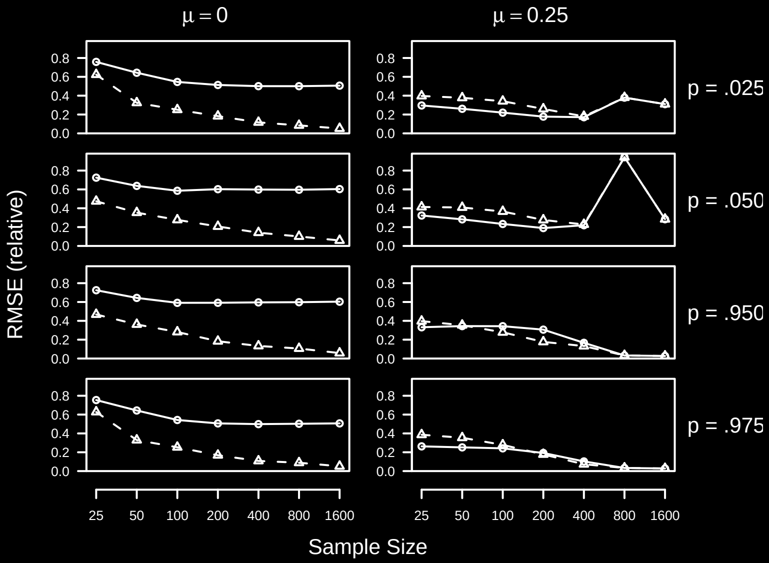
$$\Theta_J = \{\beta \in \mathbb{R}^p : \beta^{(j)} = 0 \text{ for all } j \in J,\ \beta^{(j)} \neq 0 \text{ for all } j \notin J\}$$

- $g_n(x, C; \beta) =$
$x + \operatorname{argmin}_u \Big\{ u^T C u + n^{-1/2} \lambda_n \sum_{j=1}^p \Big(|n^{1/2}\beta^{(j)} + (x+u)^{(j)}| - |n^{1/2}\beta^{(j)}| \Big) \Big\}$

- $g(x, C; \beta) =$
$x + \operatorname{argmin}_u \Big\{ u^T C u + \lambda_0 \sum_{j=1}^p \Big(u^{(j)} \operatorname{sign}(\beta^{(j)}) + \big|(x+u)^{(j)}\big| \, \mathrm{I}(\beta^{(j)} = 0) \Big) \Big\}$

- $\tilde{\beta}_n =$ an estimator with the "oracle" property, e.g., SCAD or adaptive-LASSO (but not the LASSO).

Simulations: Hodges' Estimator ($b = 0.5$), $a_n = n^{-1/2} \log_{10} n$



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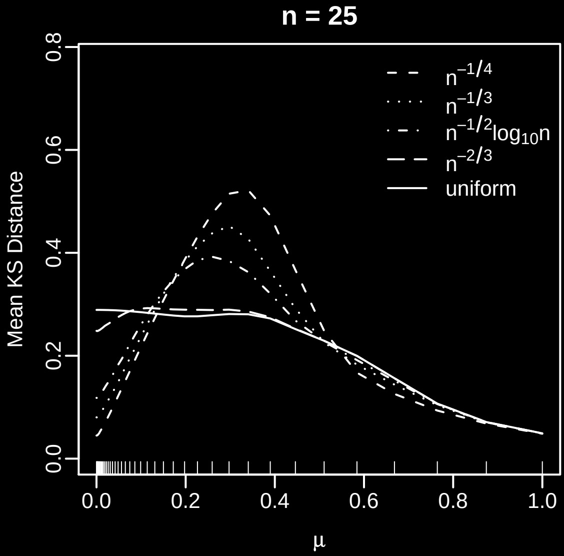

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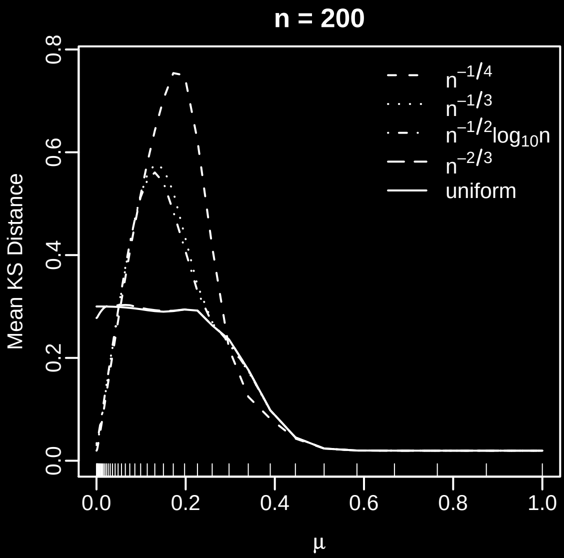

The End

No one will ever see this slide.

Proof of Corollary 1

Suppose $x_n \to x$ and $Y'_n \xrightarrow{d} Y \sim Q$.
Let $h_n(y) := g_n(x_n, y)$ and $h(y) := g(x, y)$.

$$\left.\begin{array}{l} Y'_n \xrightarrow{d} Y \\ h_n(y_n) \to h(y) \\ \qquad \text{for all}^* \ y_n \to y \end{array}\right\} \xRightarrow{\text{H.Rubin}} h_n(Y'_n) \xrightarrow{d} h(Y) \quad \Updownarrow$$

$$g_n(x_n, Y'_n) \xrightarrow{d} g(x, Y)$$

Translation: suppose $x_n \to x$ and $Q_n \to Q$. Then

$$\psi_n(x_n, Q_n) \to \psi(x, Q)$$

where

$$\begin{aligned} \psi_n(x, Q') &:= \mathscr{L}\big(g_n(x, Y')\big) \\ \psi(x, Q') &:= \mathscr{L}\big(g(x, Y')\big) \end{aligned} \quad \text{when } Y' \sim Q'$$

Return to Corollary 1

Proof of Corollary (p2)

$$\psi_n(x, Q') := \mathscr{L}\big(g_n(x, Y')\big) \qquad Y' \sim Q'$$
$$\psi(x, Q') := \mathscr{L}\big(g(x, Y')\big)$$

$$\psi_n(x_n, Q_n) \to \psi(x, Q) \qquad \text{for all}^* \ x_n \to x \text{ and } Q_n \to Q$$

Let $\mathbb{Q}_n = \mathscr{L}(Y_n | \mathscr{A}_n)$. Then

$$\left.\begin{array}{l} X_n \xrightarrow{\text{d}} X \\ \mathbb{Q}_n \xrightarrow{\text{Pr}} Q \\ \psi_n(x_n, Q_n) \to \psi(x, Q) \\ \qquad \text{for all}^* \ (x_n, Q_n) \to (x, Q) \end{array}\right\} \xRightarrow{\text{Proposition}} \psi_n(X_n, \mathbb{Q}_n) \xrightarrow{\text{d}} \psi(X, Q)$$

Return to Corollary 1

Proof of Corollary (p3)

$$\psi_n(x, Q') := \mathscr{L}\big(g_n(x, Y')\big) \qquad Y' \sim Q'$$
$$\psi(x, Q') := \mathscr{L}\big(g(x, Y')\big) \qquad Y' \sim Q'$$
$$\mathbb{Q}_n := \mathscr{L}(Y_n | \mathscr{A}_n)$$
$$\psi_n(X_n, \mathbb{Q}_n) \xrightarrow{\text{d}} \psi(X, Q) \qquad X \sim P$$

But

$$\left.\begin{aligned} X_n &\in \mathscr{A}_n \\ \mathbb{Q}_n &:= \mathscr{L}(Y_n | \mathscr{A}_n) \end{aligned}\right\} \implies \mathscr{L}\big(g_n(X_n, Y_n)\big|\mathscr{A}_n\big) = \psi_n(X_n, \mathbb{Q}_n)$$

and

$$\psi(X, Q) = \mathscr{L}\big(g(X, Y)\big|X\big), \quad X \sim P, \quad Y \sim Q, \quad X \perp\!\!\!\perp Y$$

so

$$\mathscr{L}\big(g_n(X_n, Y_n)\big|\mathscr{A}_n\big) \xrightarrow{\text{d}} \mathscr{L}\big(g(X, Y)\big|X\big), \quad X \sim P, \quad Y \sim Q, \quad X \perp\!\!\!\perp Y$$

□

Return to Corollary 1