Billingsley (3rd ed), Exercise 2.3: (a) Suppose that $\Omega \in F$ and that $A, B \in F$ implies $A - B = A \cap B^c \in F$. Show that $F$ is a field.
(b) Suppose that $\Omega \in F$ and that $F$ is closed under the formation of complements and finite disjoint unions. Show that $F$ need not be a field.

Billingsley (3rd ed), Exercise 2.4: Let $F_1, F_2, \ldots$ be classes of sets in a common space $\Omega$.
(a) Suppose that $F_n$ are fields satisfying $F_n \subset F_{n+1}$. Show that $\bigcup_{n=1}^{\infty} F_n$ is a field.
(b) Suppose that $F_n$ are $\sigma$-fields satisfying $F_n \subset F_{n+1}$. Show by example that $\bigcup_{n=1}^{\infty} F_n$ need not be a $\sigma$-field.

Billingsley (3rd ed), Exercise 2.5: The field $f(\mathcal{A})$ generated by a class $\mathcal{A}$ in $\Omega$ is defined as the intersection of all fields in $\Omega$ containing $\mathcal{A}$.
(a) Show that $f(F)$ is indeed a field, that $\mathcal{A} \subset f(\mathcal{A})$ and that $f(\mathcal{A})$ is minimal in the sense that if $G$ is a field and $\mathcal{A} \subset G$, then $f(\mathcal{A}) \subset G$.
(b) Show that for nonempty $\mathcal{A}$, $f(\mathcal{A})$ is the class of sets of the form $\bigcup_{i=1}^{m} \cap_{j=1}^{n_i} A_{ij}$, where for each $i$ and $j$ either $A_{ij} \in \mathcal{A}$ or $A_{ij}^c \in \mathcal{A}$, and where the $m$ sets $\cap_{j=1}^{n_i} A_{ij}$, $1 \leq i \leq m$, are disjoint. The sets in $f(\mathcal{A})$ can thus be explicitly presented, which is not in general true of the sets in $\sigma(\mathcal{A})$.

Billingsley (3rd ed), Exercise 2.7: Let $H$ be a set lying outside $F$, where $F$ is a field [or $\sigma$-field]. Show that the field [or $\sigma$-field] generated by $F \cup \{H\}$ consists of sets of the form

$$(H \cap A) \cup (H^c \cap B), \quad A, B \in F. \quad (2.33)$$

Billingsley (3rd ed), Exercise 2.8: Suppose for each $A$ in $\mathcal{A}$ that $A^c$ is a countable union of elements of $\mathcal{A}$. The class of intervals in $[0, 1]$ has this property. Show that $\sigma(\mathcal{A})$ coincides with the smallest class over $\mathcal{A}$ that is closed under the formation of countable unions and intersections.

Billingsley (3rd ed), Exercise 2.9: Show that if $B \in \sigma(\mathcal{A})$, then there exists a countable subclass $\mathcal{A}_B$ of $\mathcal{A}$ such that $B \in \sigma(\mathcal{A}_B)$.

Billingsley (3rd ed), Exercise 2.11: A $\sigma$-field is countably generated, or separable, if it is generated by some countable class of sets.
(a) Show that the $\sigma$-field $B$ of Borel sets is countably generated.
(b) Show that the $\sigma$-field of countable and cocountable sets is countably generated if and only if $\Omega$ is countable.
(c) Suppose that $F_1$ and $F_2$ are $\sigma$-fields, $F_1 \subset F_2$, and $F_2$ is countably generated. Show by example that $F_1$ may not be countably generated.