

Supplemental Materials for “Causal Effects of Treatments for Informative Missing Data Due to Progression/Death”

Detailed Calculations for the Covariance Matrix of $SACE_{k,\cdot}(1, 0)$

We first describe the calculation for the covariance matrix for the log of $SACE_{k,1}(1, 0)$. The log of $SACE_{k,1}(1, 0)$ is

$$\begin{aligned} \log SACE_{k,1}(1, 0) &= \log P(Y_T(1) > k | A_T(0)) - \log \{1 - P(Y_T(1) > k | A_T(0))\} \\ &\quad - \log P(Y_T(0) > k | A_T(0)) + \log \{1 - P(Y_T(0) > k | A_T(0))\}. \end{aligned}$$

The derivatives of $\log SACE_k(1, 0)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\begin{aligned} \frac{\partial \log SACE_{k,1}(1, 0)}{\partial \beta_0(\text{tx})} &= \frac{\partial \log SACE_{k,1}(1, 0)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta_0(\text{tx})}, \\ \frac{\partial \log SACE_{k,1}(1, 0)}{\partial \beta(\text{tx})} &= \frac{\partial \log SACE_{k,1}(1, 0)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta(\text{tx})}. \end{aligned}$$

$\frac{\partial h_{T,\text{tx}}(k)}{\partial \beta_0(\text{tx})}$ and $\frac{\partial h_{T,\text{tx}}(k)}{\partial \beta(\text{tx})}$ are given in Lee and Daniels (2007). The derivatives of $\log SACE_{k,1}(1, 0)$

with respect to $\psi_0, \psi_1, \psi_2, h_{T,0}(k), h_{T,1}(k),$ and $h_{T,2}(k)$ are given by

$$\frac{\partial \log SACE_{k,1}(1, 0)}{\partial \psi_0} = \frac{\frac{\partial P(Y_T(1) > k | A_T(0))}{\partial \psi_0}}{P(Y_T(1) > k | A_T(0)) \{1 - P(Y_T(1) > k | A_T(0))\}} - \frac{\frac{\partial P(Y_T(0) > k | A_T(0))}{\partial \psi_0}}{P(Y_T(0) > k | A_T(0)) \{1 - P(Y_T(0) > k | A_T(0))\}}, \quad (1)$$

$$\frac{\partial \log SACE_{k,1}(1, 0)}{\partial \psi_1} = \frac{\frac{\partial P(Y_T(1) > k | A_T(0))}{\partial \psi_1}}{P(Y_T(1) > k | A_T(0)) \{1 - P(Y_T(1) > k | A_T(0))\}} - \frac{\frac{\partial P(Y_T(0) > k | A_T(0))}{\partial \psi_1}}{P(Y_T(0) > k | A_T(0)) \{1 - P(Y_T(0) > k | A_T(0))\}}, \quad (2)$$

$$\frac{\partial \log SACE_{k,1}(1, 0)}{\partial \psi_2} = \frac{\frac{\partial P(Y_T(1) > k | A_T(0))}{\partial \psi_2}}{P(Y_T(1) > k | A_T(0)) \{1 - P(Y_T(1) > k | A_T(0))\}} - \frac{\frac{\partial P(Y_T(0) > k | A_T(0))}{\partial \psi_2}}{P(Y_T(0) > k | A_T(0)) \{1 - P(Y_T(0) > k | A_T(0))\}}, \quad (3)$$

$$\frac{\partial \log SACE_{k,1}(1, 0)}{\partial h_{T,0}(k)} = - \frac{\frac{\partial P(Y_T(0) > k | A_T(0))}{\partial h_{T,0}(k)}}{P(Y_T(0) > k | A_T(0)) \{1 - P(Y_T(0) > k | A_T(0))\}}, \quad (4)$$

$$\frac{\partial \log SACE_{k,1}(1, 0)}{\partial h_{T,1}(k)} = \frac{\frac{\partial P(Y_T(1) > k | A_T(0))}{\partial h_{T,1}(k)}}{P(Y_T(1) > k | A_T(0)) \{1 - P(Y_T(1) > k | A_T(0))\}}, \quad (5)$$

$$\frac{\partial \log SACE_{k,1}(1, 0)}{\partial h_{T,2}(k)} = 0,$$

where the numerators in (1)-(5) are calculated using numerical derivatives.

We now describe the calculation for the covariance matrix for the log of $SACE_{k,2}(1, 0)$.

The log of $SACE_{k,2}(1, 0)$ is

$$\begin{aligned} \log SACE_{k,2}(1, 0) &= \log P(Y_T(1) > k | A_T(4)) - \log \{1 - P(Y_T(1) > k | A_T(4))\} \\ &\quad - \log P(Y_T(0) > k | A_T(4)) + \log \{1 - P(Y_T(0) > k | A_T(4))\}. \end{aligned}$$

The derivatives of $\log SACE_{k,2}(1, 0)$ with respect to $\beta_0(0), \beta_0(1), \beta_0(2), \beta(0), \beta(1),$ and $\beta(2)$ are, respectively, given by

$$\begin{aligned} \frac{\partial \log SACE_{k,2}(1, 0)}{\partial \beta_0(\text{tx})} &= \frac{\partial \log SACE_{k,2}(1, 0)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta_0(\text{tx})}, \\ \frac{\partial \log SACE_{k,2}(1, 0)}{\partial \beta(\text{tx})} &= \frac{\partial \log SACE_{k,2}(1, 0)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta(\text{tx})}. \end{aligned}$$

The derivatives of $\log SACE_{k,2}(1, 0)$ with respect to $\psi_0, \psi_1, \psi_2, h_{T,0}(k), h_{T,1}(k),$ and $h_{T,2}(k)$

are given by

$$\begin{aligned}
\frac{\partial \log SACE_{k,2}(1, 0)}{\partial \psi_0} &= \frac{\frac{\partial P(Y_T(1) > k | A_T(4))}{\partial \psi_0}}{P(Y_T(1) > k | A_T(4)) \{1 - P(Y_T(1) > k | A_T(4))\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0) > k | A_T(4))}{\partial \psi_0}}{P(Y_T(0) > k | A_T(4)) \{1 - P(Y_T(0) > k | A_T(4))\}}, \\
\frac{\partial \log SACE_{k,2}(1, 0)}{\partial \psi_1} &= \frac{\frac{\partial P(Y_T(1) > k | A_T(4))}{\partial \psi_1}}{P(Y_T(1) > k | A_T(4)) \{1 - P(Y_T(1) > k | A_T(4))\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0) > k | A_T(4))}{\partial \psi_1}}{P(Y_T(0) > k | A_T(4)) \{1 - P(Y_T(0) > k | A_T(4))\}}, \\
\frac{\partial \log SACE_{k,2}(1, 0)}{\partial \psi_2} &= \frac{\frac{\partial P(Y_T(1) > k | A_T(4))}{\partial \psi_2}}{P(Y_T(1) > k | A_T(4)) \{1 - P(Y_T(1) > k | A_T(4))\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0) > k | A_T(4))}{\partial \psi_2}}{P(Y_T(0) > k | A_T(4)) \{1 - P(Y_T(0) > k | A_T(4))\}}, \\
\frac{\partial \log SACE_{k,2}(1, 0)}{\partial h_{T,0}(k)} &= - \frac{\frac{\partial P(Y_T(0) > k | A_T(4))}{\partial h_{T,0}(k)}}{P(Y_T(0) > k | A_T(4)) \{1 - P(Y_T(0) > k | A_T(4))\}}, \\
\frac{\partial \log SACE_{k,2}(1, 0)}{\partial h_{T,1}(k)} &= \frac{\frac{\partial P(Y_T(1) > k | A_T(4))}{\partial h_{T,1}(k)}}{P(Y_T(1) > k | A_T(4)) \{1 - P(Y_T(1) > k | A_T(4))\}}, \\
\frac{\partial \log SACE_{k,2}(1, 0)}{\partial h_{T,2}(k)} &= 0,
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial P(Y_T(j) > k | A_T(4))}{\partial \psi_{\text{tx}}} &= \frac{\tau \frac{\partial P(Y_T(j) > k | A_T(0))}{\partial \psi_{\text{tx}}}}{\{1 + (\tau - 1)P(Y_T(j) > k | A_T(0))\}^2}, \\
\frac{\partial P(Y_T(0) > k | A_T(4))}{\partial h_{T,0}(k)} &= \frac{\tau \frac{\partial P(Y_T(0) > k | A_T(0))}{\partial h_{T,0}(k)}}{\{1 + (\tau - 1)P(Y_T(0) > k | A_T(0))\}^2}, \\
\frac{\partial P(Y_T(1) > k | A_T(4))}{\partial h_{T,1}(k)} &= \frac{\tau \frac{\partial P(Y_T(1) > k | A_T(0))}{\partial h_{T,1}(k)}}{\{1 + (\tau - 1)P(Y_T(1) > k | A_T(0))\}^2},
\end{aligned}$$

for $j = 0, 1$ and $\text{tx} = 0, 1, 2$.

We now describe the calculation for the covariance matrix for the log of $SACE_{k,3}(1, 0)$.

The log of $SACE_{k,3}(1, 0)$ is

$$\begin{aligned}
\log SACE_{k,3}(1, 0) &= \log P(Y_T(1) > k | A_T(0) \cup A_T(4)) - \log \{1 - P(Y_T(1) > k | A_T(0) \cup A_T(4))\} \\
&\quad - \log P(Y_T(0) > k | A_T(0) \cup A_T(4)) + \log \{1 - P(Y_T(0) > k | A_T(0) \cup A_T(4))\}.
\end{aligned}$$

The derivatives of $\log \text{SACE}_{k,3}(1, 0)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\begin{aligned}\frac{\partial \log \text{SACE}_{k,3}(1, 0)}{\partial \beta_0(\text{tx})} &= \frac{\partial \log \text{SACE}_{k,3}(1, 0)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta_0(\text{tx})}, \\ \frac{\partial \log \text{SACE}_{k,3}(1, 0)}{\partial \beta(\text{tx})} &= \frac{\partial \log \text{SACE}_{k,3}(1, 0)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta(\text{tx})}.\end{aligned}$$

The derivatives of $\log \text{SACE}_{k,3}(1, 0)$ with respect to ψ_0 , ψ_1 , ψ_2 , $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$ are given by

$$\begin{aligned}\frac{\partial \log \text{SACE}_{k,3}(1, 0)}{\partial \psi_0} &= \frac{\frac{\partial P(Y_T(1) > k | A_T(0) \cup A_T(4))}{\partial \psi_0}}{P(Y_T(1) > k | A_T(0) \cup A_T(4)) \{1 - P(Y_T(1) > k | A_T(0) \cup A_T(4))\}} \\ &\quad - \frac{\frac{\partial P(Y_T(0) > k | A_T(0) \cup A_T(4))}{\partial \psi_0}}{P(Y_T(0) > k | A_T(0) \cup A_T(4)) \{1 - P(Y_T(0) > k | A_T(0) \cup A_T(4))\}}, \\ \frac{\partial \log \text{SACE}_{k,3}(1, 0)}{\partial \psi_1} &= \frac{\frac{\partial P(Y_T(1) > k | A_T(0) \cup A_T(4))}{\partial \psi_1}}{P(Y_T(1) > k | A_T(0) \cup A_T(4)) \{1 - P(Y_T(1) > k | A_T(0) \cup A_T(4))\}} \\ &\quad - \frac{\frac{\partial P(Y_T(0) > k | A_T(0) \cup A_T(4))}{\partial \psi_1}}{P(Y_T(0) > k | A_T(0) \cup A_T(4)) \{1 - P(Y_T(0) > k | A_T(0) \cup A_T(4))\}}, \\ \frac{\partial \log \text{SACE}_{k,3}(1, 0)}{\partial \psi_2} &= \frac{\frac{\partial P(Y_T(1) > k | A_T(0) \cup A_T(4))}{\partial \psi_2}}{P(Y_T(1) > k | A_T(0) \cup A_T(4)) \{1 - P(Y_T(1) > k | A_T(0) \cup A_T(4))\}} \\ &\quad - \frac{\frac{\partial P(Y_T(0) > k | A_T(0) \cup A_T(4))}{\partial \psi_2}}{P(Y_T(0) > k | A_T(0) \cup A_T(4)) \{1 - P(Y_T(0) > k | A_T(0) \cup A_T(4))\}}, \\ \frac{\partial \log \text{SACE}_{k,3}(1, 0)}{\partial h_{T,0}(k)} &= - \frac{\frac{\partial P(Y_T(0) > k | A_T(0) \cup A_T(4))}{\partial h_{T,0}(k)}}{P(Y_T(0) > k | A_T(0) \cup A_T(4)) \{1 - P(Y_T(0) > k | A_T(0) \cup A_T(4))\}}, \\ \frac{\partial \log \text{SACE}_{k,3}(1, 0)}{\partial h_{T,1}(k)} &= \frac{\frac{\partial P(Y_T(1) > k | A_T(0) \cup A_T(4))}{\partial h_{T,1}(k)}}{P(Y_T(1) > k | A_T(0) \cup A_T(4)) \{1 - P(Y_T(1) > k | A_T(0) \cup A_T(4))\}}, \\ \frac{\partial \log \text{SACE}_{k,3}(1, 0)}{\partial h_{T,2}(k)} &= 0,\end{aligned}$$

with

$$\begin{aligned}
\frac{\partial P(A_T(0))}{\partial \psi_0} &= E_w \left[\frac{\partial g_T(0, w)}{\partial \psi_0} \{p_T^*(2, w) - p_T^*(1, w) + 1\} + g_T(0, w) \left\{ \frac{\partial p_T^*(2, w)}{\partial \psi_0} - \frac{\partial p_T^*(1, w)}{\partial \psi_0} \right\} \right], \\
\frac{\partial P(A_T(4))}{\partial \psi_0} &= E_w \left\{ \frac{\partial p_T^*(1, w)}{\partial \psi_0} + p_T^*(1, w) \frac{\partial g_T(0, w)}{\partial \psi_0} \right\} - \frac{\partial P(A_T(0))}{\partial \psi_0}, \\
\frac{\partial P(A_T(0) \cup A_T(4))}{\partial \psi_0} &= \frac{\partial P(A_T(0))}{\partial \psi_0} + \frac{\partial P(A_T(4))}{\partial \psi_0}, \\
\frac{\partial P(A_T(0))}{\partial \psi_1} &= -E_w \left\{ g_T(0, w) \frac{\partial p_T^*(1, w)}{\partial \psi_1} \right\}, \\
\frac{\partial P(A_T(4))}{\partial \psi_1} &= 2E_w \left\{ g_T(0, w) \frac{\partial p_T^*(1, w)}{\partial \psi_1} \right\}, \\
\frac{\partial P(A_T(0) \cup A_T(4))}{\partial \psi_1} &= \frac{\partial P(A_T(0))}{\partial \psi_1} + \frac{\partial P(A_T(4))}{\partial \psi_1}, \\
\frac{\partial P(A_T(0))}{\partial \psi_2} &= E_w \left\{ g_T(0, w) \frac{\partial p_T^*(2, w)}{\partial \psi_2} \right\}, \\
\frac{\partial P(A_T(4))}{\partial \psi_2} &= -\frac{\partial P(A_T(0))}{\partial \psi_2}.
\end{aligned}$$

Detailed Calculations for the Covariance Matrix of $SACE_{k,\cdot}(2, 0)$

The log of $SACE_{k,1}(2, 0)$ is

$$\begin{aligned}
\log SACE_{k,1}(2, 0) &= \log P(Y_T(2) > k | A_T(0)) - \log \{1 - P(Y_T(2) > k | A_T(0))\} \\
&\quad - \log P(Y_T(0) > k | A_T(0)) + \log \{1 - P(Y_T(0) > k | A_T(0))\}.
\end{aligned}$$

Similar to the calculation of derivatives in $SACE_{k,1}(1, 0)$, the derivatives of $\log SACE_{k,1}(2, 0)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\begin{aligned}
\frac{\partial \log SACE_{k,1}(2, 0)}{\partial \beta_0(\text{tx})} &= \frac{\partial \log SACE_{k,1}(2, 0)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta_0(\text{tx})}, \\
\frac{\partial \log SACE_{k,1}(2, 0)}{\partial \beta(\text{tx})} &= \frac{\partial \log SACE_{k,1}(2, 0)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta(\text{tx})}.
\end{aligned}$$

The derivatives of $\log SACE_{k,1}(2, 0)$ with respect to ψ_0 , ψ_1 , ψ_2 , $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$

are, respectively, given by

$$\begin{aligned} & \frac{\partial \log SACE_{k,1}(2,0)}{\partial \psi_0} \\ = & \frac{\frac{\partial P(Y_T(2) > k | A_T(0))}{\partial \psi_0}}{P(Y_T(2) > k | A_T(0)) \{1 - P(Y_T(2) > k | A_T(0))\}} - \frac{\frac{\partial P(Y_T(0) > k | A_T(0))}{\partial \psi_0}}{P(Y_T(0) > k | A_T(0)) \{1 - P(Y_T(0) > k | A_T(0))\}}, \end{aligned} \quad (6)$$

$$\begin{aligned} & \frac{\partial \log SACE_{k,1}(2,0)}{\partial \psi_1} \\ = & \frac{\frac{\partial P(Y_T(2) > k | A_T(0))}{\partial \psi_1}}{P(Y_T(2) > k | A_T(0)) \{1 - P(Y_T(2) > k | A_T(0))\}} - \frac{\frac{\partial P(Y_T(0) > k | A_T(0))}{\partial \psi_1}}{P(Y_T(0) > k | A_T(0)) \{1 - P(Y_T(0) > k | A_T(0))\}}, \end{aligned} \quad (7)$$

$$\begin{aligned} & \frac{\partial \log SACE_{k,1}(2,0)}{\partial \psi_2} \\ = & \frac{\frac{\partial P(Y_T(2) > k | A_T(0))}{\partial \psi_2}}{P(Y_T(2) > k | A_T(0)) \{1 - P(Y_T(2) > k | A_T(0))\}} - \frac{\frac{\partial P(Y_T(0) > k | A_T(0))}{\partial \psi_2}}{P(Y_T(0) > k | A_T(0)) \{1 - P(Y_T(0) > k | A_T(0))\}}, \end{aligned} \quad (8)$$

$$\frac{\partial \log SACE_{k,1}(2,0)}{\partial h_{T,0}(k)} = - \frac{\frac{\partial P(Y_T(0) > k | A_T(0))}{\partial h_{T,0}(k)}}{P(Y_T(0) > k | A_T(0)) \{1 - P(Y_T(0) > k | A_T(0))\}}, \quad (9)$$

$$\frac{\partial \log SACE_{k,1}(2,0)}{\partial h_{T,1}(k)} = 0,$$

$$\frac{\partial \log SACE_{k,1}(2,0)}{\partial h_{T,2}(k)} = \frac{\frac{\partial P(Y_T(2) > k | A_T(0))}{\partial h_{T,2}(k)}}{P(Y_T(2) > k | A_T(0)) \{1 - P(Y_T(2) > k | A_T(0))\}}. \quad (10)$$

where the numerator parts in (6)-(10) are calculated using numerical derivatives. We now describe the calculation for the covariance matrix for the log of $SACE_{k,2}(2,0)$. The log of $SACE_{k,2}(2,0)$ is

$$\begin{aligned} \log SACE_{k,2}(2,0) &= \log P(Y_T(2) > k | A_T(5)) - \log \{1 - P(Y_T(2) > k | A_T(5))\} \\ &\quad - \log P(Y_T(0) > k | A_T(5)) + \log \{1 - P(Y_T(0) > k | A_T(5))\}. \end{aligned}$$

The derivatives of $\log SACE_{k,2}(2,0)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\begin{aligned} \frac{\partial \log SACE_{k,2}(2,0)}{\partial \beta_0(\text{tx})} &= \frac{\partial \log SACE_{k,2}(2,0)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta_0(\text{tx})}, \\ \frac{\partial \log SACE_{k,2}(2,0)}{\partial \beta(\text{tx})} &= \frac{\partial \log SACE_{k,2}(2,0)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta(\text{tx})}. \end{aligned}$$

The derivatives of $\log SACE_{k,2}(2,0)$ with respect to ψ_0 , ψ_1 , ψ_2 , $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$

are given by

$$\begin{aligned}
\frac{\partial \log SACE_{k,2}(2, 0)}{\partial \psi_0} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(5))}{\partial \psi_0}}{P(Y_T(2) > k | A_T(5)) \{1 - P(Y_T(2) > k | A_T(5))\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0) > k | A_T(5))}{\partial \psi_0}}{P(Y_T(0) > k | A_T(5)) \{1 - P(Y_T(0) > k | A_T(5))\}}, \\
\frac{\partial \log SACE_{k,2}(2, 0)}{\partial \psi_1} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(5))}{\partial \psi_1}}{P(Y_T(2) > k | A_T(5)) \{1 - P(Y_T(2) > k | A_T(5))\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0) > k | A_T(5))}{\partial \psi_1}}{P(Y_T(0) > k | A_T(5)) \{1 - P(Y_T(0) > k | A_T(5))\}}, \\
\frac{\partial \log SACE_{k,2}(2, 0)}{\partial \psi_2} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(5))}{\partial \psi_2}}{P(Y_T(2) > k | A_T(5)) \{1 - P(Y_T(2) > k | A_T(5))\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0) > k | A_T(5))}{\partial \psi_2}}{P(Y_T(0) > k | A_T(5)) \{1 - P(Y_T(0) > k | A_T(5))\}}, \\
\frac{\partial \log SACE_{k,2}(2, 0)}{\partial h_{T,0}(k)} &= - \frac{\frac{\partial P(Y_T(0) > k | A_T(5))}{\partial h_{T,0}(k)}}{P(Y_T(0) > k | A_T(5)) \{1 - P(Y_T(0) > k | A_T(5))\}}, \\
\frac{\partial \log SACE_{k,2}(2, 0)}{\partial h_{T,1}(k)} &= 0, \\
\frac{\partial \log SACE_{k,2}(2, 0)}{\partial h_{T,2}(k)} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(5))}{\partial h_{T,2}(k)}}{P(Y_T(2) > k | A_T(5)) \{1 - P(Y_T(2) > k | A_T(5))\}},
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial P(Y_T(j) > k | A_T(5))}{\partial \psi_{\text{tx}}} &= \frac{\tau \frac{\partial P(Y_T(j) > k | A_T(0))}{\partial \psi_{\text{tx}}}}{\{1 + (\tau - 1)P(Y_T(j) > k | A_T(0))\}^2}, \\
\frac{\partial P(Y_T(0) > k | A_T(5))}{\partial h_{T,0}(k)} &= \frac{\tau \frac{\partial P(Y_T(0) > k | A_T(0))}{\partial h_{T,0}(k)}}{\{1 + (\tau - 1)P(Y_T(0) > k | A_T(0))\}^2}, \\
\frac{\partial P(Y_T(2) > k | A_T(5))}{\partial h_{T,2}(k)} &= \frac{\tau \frac{\partial P(Y_T(2) > k | A_T(0))}{\partial h_{T,2}(k)}}{\{1 + (\tau - 1)P(Y_T(2) > k | A_T(0))\}^2},
\end{aligned}$$

for $j = 0, 2$ and $\text{tx} = 0, 1, 2$.

We now describe the calculation for the covariance matrix for the log of $SACE_{k,3}(2, 0)$.

The log of $SACE_{k,3}(2, 0)$ is

$$\begin{aligned}
\log SACE_{k,3}(2, 0) &= \log P(Y_T(2) > k | A_T(0) \cup A_T(5)) - \log \{1 - P(Y_T(2) > k | A_T(0) \cup A_T(5))\} \\
&\quad - \log P(Y_T(0) > k | A_T(0) \cup A_T(5)) + \log \{1 - P(Y_T(0) > k | A_T(0) \cup A_T(5))\}.
\end{aligned}$$

The derivatives of $\log \text{SACE}_{k,3}(2, 0)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\begin{aligned}\frac{\partial \log \text{SACE}_{k,3}(2, 0)}{\partial \beta_0(\text{tx})} &= \frac{\partial \log \text{SACE}_{k,3}(2, 0)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta_0(\text{tx})}, \\ \frac{\partial \log \text{SACE}_{k,3}(2, 0)}{\partial \beta(\text{tx})} &= \frac{\partial \log \text{SACE}_{k,3}(2, 0)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta(\text{tx})}.\end{aligned}$$

The derivatives of $\log \text{SACE}_{k,3}(2, 0)$ with respect to ψ_0 , ψ_1 , ψ_2 , $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$ are given by

$$\begin{aligned}\frac{\partial \log \text{SACE}_{k,3}(2, 0)}{\partial \psi_0} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(0) \cup A_T(5))}{\partial \psi_0}}{P(Y_T(2) > k | A_T(0) \cup A_T(5)) \{1 - P(Y_T(2) > k | A_T(0) \cup A_T(5))\}} \\ &\quad - \frac{\frac{\partial P(Y_T(0) > k | A_T(0) \cup A_T(5))}{\partial \psi_0}}{P(Y_T(0) > k | A_T(0) \cup A_T(5)) \{1 - P(Y_T(0) > k | A_T(0) \cup A_T(5))\}}, \\ \frac{\partial \log \text{SACE}_{k,3}(2, 0)}{\partial \psi_1} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(0) \cup A_T(5))}{\partial \psi_1}}{P(Y_T(2) > k | A_T(0) \cup A_T(5)) \{1 - P(Y_T(1) > k | A_T(0) \cup A_T(5))\}} \\ &\quad - \frac{\frac{\partial P(Y_T(0) > k | A_T(0) \cup A_T(5))}{\partial \psi_1}}{P(Y_T(0) > k | A_T(0) \cup A_T(5)) \{1 - P(Y_T(0) > k | A_T(0) \cup A_T(5))\}}, \\ \frac{\partial \log \text{SACE}_{k,3}(2, 0)}{\partial \psi_2} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(0) \cup A_T(5))}{\partial \psi_2}}{P(Y_T(2) > k | A_T(0) \cup A_T(5)) \{1 - P(Y_T(2) > k | A_T(0) \cup A_T(5))\}} \\ &\quad - \frac{\frac{\partial P(Y_T(0) > k | A_T(0) \cup A_T(5))}{\partial \psi_2}}{P(Y_T(0) > k | A_T(0) \cup A_T(5)) \{1 - P(Y_T(0) > k | A_T(0) \cup A_T(5))\}}, \\ \frac{\partial \log \text{SACE}_{k,3}(2, 0)}{\partial h_{T,0}(k)} &= - \frac{\frac{\partial P(Y_T(2) > k | A_T(0) \cup A_T(5))}{\partial h_{T,0}(k)}}{P(Y_T(0) > k | A_T(0) \cup A_T(5)) \{1 - P(Y_T(0) > k | A_T(0) \cup A_T(5))\}}, \\ \frac{\partial \log \text{SACE}_{k,3}(2, 0)}{\partial h_{T,1}(k)} &= 0, \\ \frac{\partial \log \text{SACE}_{k,3}(2, 0)}{\partial h_{T,2}(k)} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(0) \cup A_T(5))}{\partial h_{T,2}(k)}}{P(Y_T(2) > k | A_T(0) \cup A_T(5)) \{1 - P(Y_T(2) > k | A_T(0) \cup A_T(5))\}},\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial P(Y_T(0) > k | A_T(0) \cup A_T(5))}{\partial \psi_0} &= \frac{1}{P(A_T(0) \cup A_T(5))} \left\{ \frac{\partial P(Y_T(0) > k | A_T(0))}{\partial \psi_0} P(A_T(0)) + P(Y_T(0) > k | A_T(0)) \frac{\partial P(A_T(0))}{\partial \psi_0} \right. \\
&\quad \left. + \frac{\partial P(Y_T(0) > k | A_T(5))}{\partial \psi_0} P(A_T(5)) + P(Y_T(0) > k | A_T(5)) \frac{\partial P(A_T(5))}{\partial \psi_0} \right\} \\
&\quad - \frac{P(Y_T(0) > k | A_T(0)) P(A_T(0)) + P(Y_T(0) > k | A_T(5)) P(A_T(5))}{\{P(A_T(0) \cup A_T(5))\}^2} \frac{\partial P(A_T(0) \cup A_T(5))}{\partial \psi_0}, \\
\frac{\partial P(Y_T(2) > k | A_T(0) \cup A_T(5))}{\partial \psi_0} &= \frac{1}{P(A_T(0) \cup A_T(5))} \left\{ \frac{\partial P(Y_T(2) > k | A_T(0))}{\partial \psi_0} P(A_T(0)) + P(Y_T(2) > k | A_T(0)) \frac{\partial P(A_T(0))}{\partial \psi_0} \right. \\
&\quad \left. + \frac{\partial P(Y_T(2) > k | A_T(5))}{\partial \psi_0} P(A_T(5)) + P(Y_T(2) > k | A_T(5)) \frac{\partial P(A_T(5))}{\partial \psi_0} \right\} \\
&\quad - \frac{P(Y_T(2) > k | A_T(0)) P(A_T(0)) + P(Y_T(2) > k | A_T(5)) P(A_T(5))}{\{P(A_T(0) \cup A_T(5))\}^2} \frac{\partial P(A_T(0) \cup A_T(5))}{\partial \psi_0}, \\
\frac{\partial P(Y_T(0) > k | A_T(0) \cup A_T(5))}{\partial \psi_1} &= \frac{1}{P(A_T(0) \cup A_T(5))} \left\{ \frac{\partial P(Y_T(0) > k | A_T(0))}{\partial \psi_1} P(A_T(0)) + P(Y_T(0) > k | A_T(0)) \frac{\partial P(A_T(0))}{\partial \psi_1} \right. \\
&\quad \left. + \frac{\partial P(Y_T(0) > k | A_T(5))}{\partial \psi_1} P(A_T(5)) + P(Y_T(0) > k | A_T(5)) \frac{\partial P(A_T(5))}{\partial \psi_1} \right\}, \\
\frac{\partial P(Y_T(2) > k | A_T(0) \cup A_T(5))}{\partial \psi_1} &= \frac{1}{P(A_T(0) \cup A_T(5))} \left\{ \frac{\partial P(Y_T(2) > k | A_T(0))}{\partial \psi_1} P(A_T(0)) + P(Y_T(2) > k | A_T(0)) \frac{\partial P(A_T(0))}{\partial \psi_1} \right. \\
&\quad \left. + \frac{\partial P(Y_T(2) > k | A_T(5))}{\partial \psi_1} P(A_T(5)) + P(Y_T(2) > k | A_T(5)) \frac{\partial P(A_T(5))}{\partial \psi_1} \right\}, \\
\frac{\partial P(Y_T(0) > k | A_T(0) \cup A_T(5))}{\partial \psi_2} &= \frac{1}{P(A_T(0) \cup A_T(5))} \left\{ \frac{\partial P(Y_T(0) > k | A_T(0))}{\partial \psi_2} P(A_T(0)) + P(Y_T(0) > k | A_T(0)) \frac{\partial P(A_T(0))}{\partial \psi_2} \right. \\
&\quad \left. + \frac{\partial P(Y_T(0) > k | A_T(5))}{\partial \psi_2} P(A_T(5)) \right\} \\
&\quad - \frac{P(Y_T(0) > k | A_T(0)) P(A_T(0)) + P(Y_T(0) > k | A_T(5)) P(A_T(5))}{\{P(A_T(0) \cup A_T(5))\}^2} \frac{\partial P(A_T(0) \cup A_T(5))}{\partial \psi_2}, \\
\frac{\partial P(Y_T(2) > k | A_T(0) \cup A_T(5))}{\partial \psi_2} &= \frac{1}{P(A_T(0) \cup A_T(5))} \left\{ \frac{\partial P(Y_T(2) > k | A_T(0))}{\partial \psi_2} P(A_T(0)) + P(Y_T(2) > k | A_T(0)) \frac{\partial P(A_T(0))}{\partial \psi_2} \right. \\
&\quad \left. + \frac{\partial P(Y_T(2) > k | A_T(5))}{\partial \psi_2} P(A_T(5)) \right\} \\
&\quad - \frac{P(Y_T(2) > k | A_T(0)) P(A_T(0)) + P(Y_T(2) > k | A_T(5)) P(A_T(5))}{\{P(A_T(0) \cup A_T(5))\}^2} \frac{\partial P(A_T(0) \cup A_T(5))}{\partial \psi_2}, \\
\frac{\partial P(Y_T(0) > k | A_T(0) \cup A_T(5))}{\partial h_{T,0}} &= \frac{1}{P(A_T(0) \cup A_T(5))} \left\{ \frac{\partial P(Y_T(0) > k | A_T(0))}{\partial h_{T,0}(k)} P(A_T(0)) + \frac{\partial P(Y_T(0) > k | A_T(5))}{\partial h_{T,0}(k)} P(A_T(5)) \right\}, \\
\frac{\partial P(Y_T(2) > k | A_T(0) \cup A_T(5))}{\partial h_{T,2}} &= \frac{1}{P(A_T(0) \cup A_T(5))} \left\{ \frac{\partial P(Y_T(2) > k | A_T(0))}{\partial h_{T,2}(k)} P(A_T(0)) + \frac{\partial P(Y_T(2) > k | A_T(5))}{\partial h_{T,2}(k)} P(A_T(5)) \right\},
\end{aligned}$$

with

$$\begin{aligned}
\frac{\partial P(A_T(5))}{\partial \psi_0} &= E_w \left\{ \frac{\partial g_T(0, w)}{\partial \psi_0} (1 - p_T^*(1, w)) - g_T^*(0, w) \frac{\partial p_T^*(1, w)}{\partial \psi_0} \right\}, \\
\frac{\partial P(A_T(0) \cup A_T(5))}{\partial \psi_0} &= \frac{\partial P(A_T(0))}{\partial \psi_0} + \frac{\partial P(A_T(5))}{\partial \psi_0}, \\
\frac{\partial P(A_T(5))}{\partial \psi_1} &= -E_w \left\{ g_T(0, w) \frac{\partial p_T^*(1, w)}{\partial \psi_1} \right\}, \\
\frac{\partial P(A_T(0) \cup A_T(5))}{\partial \psi_2} &= \frac{\partial P(A_T(0))}{\partial \psi_2}.
\end{aligned}$$

Detailed Calculations for the Covariance Matrix of $SACE_{k,\cdot}(2, 1)$

The log of $SACE_{k,1}(2, 1)$ is

$$\begin{aligned} \log SACE_{k,1}(2, 1) &= \log P(Y_T(2) > k|A_T(0)) - \log \{1 - P(Y_T(2) > k|A_T(0))\} \\ &\quad - \log P(Y_T(1) > k|A_T(0)) + \log \{1 - P(Y_T(1) > k|A_T(0))\}. \end{aligned}$$

The derivatives of $\log SACE_{k,1}(2, 1)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\begin{aligned} \frac{\partial \log SACE_{k,1}(2, 1)}{\partial \beta_0(\text{tx})} &= \frac{\partial \log SACE_{k,1}(2, 1)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta_0(\text{tx})}, \\ \frac{\partial \log SACE_{k,1}(2, 1)}{\partial \beta(\text{tx})} &= \frac{\partial \log SACE_{k,1}(2, 1)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta(\text{tx})}. \end{aligned}$$

The derivatives of $\log SACE_{k,1}(2, 1)$ with respect to ψ_0 , ψ_1 , ψ_2 , $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$ are given by

$$\begin{aligned} &\frac{\partial \log SACE_{k,1}(2, 1)}{\partial \psi_0} \\ &= \frac{\frac{\partial P(Y_T(2) > k|A_T(0))}{\partial \psi_0}}{P(Y_T(2) > k|A_T(0)) \{1 - P(Y_T(2) > k|A_T(0))\}} - \frac{\frac{\partial P(Y_T(1) > k|A_T(0))}{\partial \psi_0}}{P(Y_T(1) > k|A_T(0)) \{1 - P(Y_T(1) > k|A_T(0))\}}, \\ &\frac{\partial \log SACE_{k,1}(2, 1)}{\partial \psi_1} \\ &= \frac{\frac{\partial P(Y_T(2) > k|A_T(0))}{\partial \psi_1}}{P(Y_T(2) > k|A_T(0)) \{1 - P(Y_T(2) > k|A_T(0))\}} - \frac{\frac{\partial P(Y_T(1) > k|A_T(0))}{\partial \psi_1}}{P(Y_T(1) > k|A_T(0)) \{1 - P(Y_T(1) > k|A_T(0))\}}, \\ &\frac{\partial \log SACE_{k,1}(2, 1)}{\partial \psi_2} \\ &= \frac{\frac{\partial P(Y_T(2) > k|A_T(0))}{\partial \psi_2}}{P(Y_T(2) > k|A_T(0)) \{1 - P(Y_T(2) > k|A_T(0))\}} - \frac{\frac{\partial P(Y_T(1) > k|A_T(0))}{\partial \psi_2}}{P(Y_T(1) > k|A_T(0)) \{1 - P(Y_T(1) > k|A_T(0))\}}, \\ &\frac{\partial \log SACE_{k,1}(2, 1)}{\partial h_{T,0}(k)} = 0, \\ &\frac{\partial \log SACE_{k,1}(2, 1)}{\partial h_{T,1}(k)} = -\frac{\frac{\partial P(Y_T(1) > k|A_T(0))}{\partial h_{T,1}(k)}}{P(Y_T(1) > k|A_T(0)) \{1 - P(Y_T(1) > k|A_T(0))\}}, \\ &\frac{\partial \log SACE_{k,1}(2, 1)}{\partial h_{T,2}(k)} = \frac{\frac{\partial P(Y_T(2) > k|A_T(0))}{\partial h_{T,2}(k)}}{P(Y_T(2) > k|A_T(0)) \{1 - P(Y_T(2) > k|A_T(0))\}}. \end{aligned}$$

We now describe the calculation for the covariance matrix for the log of $SACE_{k,2}(2, 1)$.

The log of $SACE_{k,2}(2, 1)$ is

$$\begin{aligned} \log SACE_{k,2}(2, 1) &= \log P(Y_T(2) > k|A_T(1)) - \log \{1 - P(Y_T(2) > k|A_T(1))\} \\ &\quad - \log P(Y_T(1) > k|A_T(1)) + \log \{1 - P(Y_T(1) > k|A_T(1))\}. \end{aligned}$$

The derivatives of $\log \text{SACE}_{k,2}(2, 1)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\begin{aligned}\frac{\partial \log \text{SACE}_{k,2}(2, 1)}{\partial \beta_0(\text{tx})} &= \frac{\partial \log \text{SACE}_{k,2}(2, 1)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta_0(\text{tx})}, \\ \frac{\partial \log \text{SACE}_{k,2}(2, 1)}{\partial \beta(\text{tx})} &= \frac{\partial \log \text{SACE}_{k,2}(2, 1)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta(\text{tx})}.\end{aligned}$$

The derivatives of $\log \text{SACE}_{k,2}(2, 1)$ with respect to ψ_0 , ψ_1 , ψ_2 , $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$ are given by

$$\begin{aligned}\frac{\partial \log \text{SACE}_{k,2}(2, 1)}{\partial \psi_0} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(1))}{\partial \psi_0}}{P(Y_T(2) > k | A_T(1)) \{1 - P(Y_T(2) > k | A_T(1))\}} \\ &\quad - \frac{\frac{\partial P(Y_T(1) > k | A_T(1))}{\partial \psi_0}}{P(Y_T(1) > k | A_T(1)) \{1 - P(Y_T(1) > k | A_T(1))\}}, \\ \frac{\partial \log \text{SACE}_{k,2}(2, 1)}{\partial \psi_1} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(1))}{\partial \psi_1}}{P(Y_T(2) > k | A_T(1)) \{1 - P(Y_T(2) > k | A_T(1))\}} \\ &\quad - \frac{\frac{\partial P(Y_T(1) > k | A_T(1))}{\partial \psi_1}}{P(Y_T(1) > k | A_T(1)) \{1 - P(Y_T(1) > k | A_T(1))\}}, \\ \frac{\partial \log \text{SACE}_{k,2}(2, 1)}{\partial \psi_2} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(1))}{\partial \psi_2}}{P(Y_T(2) > k | A_T(1)) \{1 - P(Y_T(2) > k | A_T(1))\}} \\ &\quad - \frac{\frac{\partial P(Y_T(1) > k | A_T(1))}{\partial \psi_2}}{P(Y_T(1) > k | A_T(1)) \{1 - P(Y_T(1) > k | A_T(1))\}}, \\ \frac{\partial \log \text{SACE}_{k,2}(2, 1)}{\partial h_{T,0}(k)} &= 0, \\ \frac{\partial \log \text{SACE}_{k,2}(2, 1)}{\partial h_{T,1}(k)} &= - \frac{\frac{\partial P(Y_T(1) > k | A_T(1))}{\partial h_{T,1}(k)}}{P(Y_T(1) > k | A_T(1)) \{1 - P(Y_T(1) > k | A_T(1))\}}, \\ \frac{\partial \log \text{SACE}_{k,2}(2, 1)}{\partial h_{T,2}(k)} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(1))}{\partial h_{T,2}(k)}}{P(Y_T(2) > k | A_T(1)) \{1 - P(Y_T(2) > k | A_T(1))\}},\end{aligned}$$

where

$$\begin{aligned}\frac{\partial P(Y_T(j) > k | A_T(1))}{\partial \psi_{\text{tx}}} &= \frac{\tau \frac{\partial P(Y_T(j) > k | A_T(0))}{\partial \psi_{\text{tx}}}}{\{1 + (\tau - 1)P(Y_T(j) > k | A_T(0))\}^2}, \\ \frac{\partial P(Y_T(1) > k | A_T(1))}{\partial h_{T,1}(k)} &= \frac{\tau \frac{\partial P(Y_T(1) > k | A_T(0))}{\partial h_{T,1}(k)}}{\{1 + (\tau - 1)P(Y_T(1) > k | A_T(0))\}^2}, \\ \frac{\partial P(Y_T(2) > k | A_T(1))}{\partial h_{T,2}(k)} &= \frac{\tau \frac{\partial P(Y_T(2) > k | A_T(0))}{\partial h_{T,2}(k)}}{\{1 + (\tau - 1)P(Y_T(2) > k | A_T(0))\}^2},\end{aligned}$$

for $j = 1, 2$ and $\text{tx} = 0, 1, 2$.

We finally describe the calculation for the covariance matrix for the log of $SACE_{k,3}(2, 1)$.

The log of $SACE_{k,3}(2, 1)$ is

$$\begin{aligned}\log SACE_{k,3}(2, 1) &= \log P(Y_T(2) > k | A_T(0) \cup A_T(1)) - \log \{1 - P(Y_T(2) > k | A_T(0) \cup A_T(1))\} \\ &\quad - \log P(Y_T(1) > k | A_T(0) \cup A_T(1)) + \log \{1 - P(Y_T(1) > k | A_T(0) \cup A_T(1))\}.\end{aligned}$$

The derivatives of $\log SACE_{k,3}(2, 1)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\begin{aligned}\frac{\partial \log SACE_{k,3}(2, 1)}{\partial \beta_0(\text{tx})} &= \frac{\partial \log SACE_{k,3}(2, 1)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta_0(\text{tx})}, \\ \frac{\partial \log SACE_{k,3}(2, 1)}{\partial \beta(\text{tx})} &= \frac{\partial \log SACE_{k,3}(2, 1)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta(\text{tx})}.\end{aligned}$$

The derivatives of $\log SACE_{k,3}(2, 1)$ with respect to ψ_0 , ψ_1 , ψ_2 , $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$

are given by

$$\begin{aligned}
\frac{\partial \log SACE_{k,3}(2,1)}{\partial \psi_0} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(0) \cup A_T(1))}{\partial \psi_0}}{P(Y_T(2) > k | A_T(0) \cup A_T(1)) \{1 - P(Y_T(2) > k | A_T(0) \cup A_T(1))\}} \\
&\quad - \frac{\frac{\partial P(Y_T(1) > k | A_T(0) \cup A_T(1))}{\partial \psi_0}}{P(Y_T(1) > k | A_T(0) \cup A_T(1)) \{1 - P(Y_T(1) > k | A_T(0) \cup A_T(1))\}}, \\
\frac{\partial \log SACE_{k,3}(2,1)}{\partial \psi_1} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(0) \cup A_T(1))}{\partial \psi_1}}{P(Y_T(2) > k | A_T(0) \cup A_T(1)) \{1 - P(Y_T(2) > k | A_T(0) \cup A_T(1))\}} \\
&\quad - \frac{\frac{\partial P(Y_T(1) > k | A_T(0) \cup A_T(1))}{\partial \psi_1}}{P(Y_T(1) > k | A_T(0) \cup A_T(1)) \{1 - P(Y_T(1) > k | A_T(0) \cup A_T(1))\}}, \\
\frac{\partial \log SACE_{k,3}(2,1)}{\partial \psi_2} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(0) \cup A_T(1))}{\partial \psi_2}}{P(Y_T(2) > k | A_T(0) \cup A_T(1)) \{1 - P(Y_T(2) > k | A_T(0) \cup A_T(1))\}} \\
&\quad - \frac{\frac{\partial P(Y_T(1) > k | A_T(0) \cup A_T(1))}{\partial \psi_2}}{P(Y_T(1) > k | A_T(0) \cup A_T(1)) \{1 - P(Y_T(1) > k | A_T(0) \cup A_T(1))\}}, \\
\frac{\partial \log SACE_{k,3}(2,1)}{\partial h_{T,0}(k)} &= 0, \\
\frac{\partial \log SACE_{k,3}(2,1)}{\partial h_{T,1}(k)} &= - \frac{\frac{\partial P(Y_T(1) > k | A_T(0) \cup A_T(1))}{\partial h_{T,1}(k)}}{P(Y_T(1) > k | A_T(0) \cup A_T(1)) \{1 - P(Y_T(1) > k | A_T(0) \cup A_T(1))\}}, \\
\frac{\partial \log SACE_{k,3}(2,1)}{\partial h_{T,2}(k)} &= \frac{\frac{\partial P(Y_T(2) > k | A_T(0) \cup A_T(1))}{\partial h_{T,2}(k)}}{P(Y_T(2) > k | A_T(0) \cup A_T(1)) \{1 - P(Y_T(2) > k | A_T(0) \cup A_T(1))\}},
\end{aligned}$$

with

$$\begin{aligned}
\frac{\partial P(A_T(1))}{\partial \psi_0} &= -E_w \left\{ \frac{\partial g_T(0, w)}{\partial \psi_0} p_T^*(2, w) + g_T(0, w) \frac{\partial p_T^*(2, w)}{\partial \psi_0} \right\} - \frac{\partial P(A_T(2))}{\partial \psi_0}, \\
\frac{\partial P(A_T(2))}{\partial \psi_0} &= E_w \left[-\frac{\partial q_T^*(w)}{\partial \psi_0} \{1 - g_T(0, w) - g_T(1, w) + p_T^*(1, w)g_T(0, w)\} \right. \\
&\quad \left. + (1 - q_T^*(w)) \left\{ -\frac{\partial g_T(0, w)}{\partial \psi_0} + \frac{\partial p_T^*(1, w)}{\partial \psi_0} g_T(0, w) + p_T^*(1, w) \frac{\partial g_T(0, w)}{\partial \psi_0} \right\} \right], \\
\frac{\partial P(A_T(0) \cup A_T(1))}{\partial \psi_0} &= \frac{\partial P(A_T(0))}{\partial \psi_0} + \frac{\partial P(A_T(1))}{\partial \psi_0}, \\
\frac{\partial P(A_T(1))}{\partial \psi_1} &= -E_w \left[\frac{\partial q_T^*(w)}{\partial \psi_1} \{1 - g_T(0, w) - g_T(1, w) + p_T^*(1, w)g_T(0, w)\} \right. \\
&\quad \left. + (1 - q_T^*(w)) \left\{ \frac{\partial g_T(1, w)}{\partial \psi_1} - g_T(0, w) \frac{\partial p_T^*(1, w)}{\partial \psi_1} \right\} \right], \\
\frac{\partial P(A_T(0) \cup A_T(1))}{\partial \psi_1} &= \frac{\partial P(A_T(0))}{\partial \psi_1} + \frac{\partial P(A_T(1))}{\partial \psi_1}, \\
\frac{\partial P(A_T(1))}{\partial \psi_2} &= E_w \left\{ \frac{\partial g_T(2, w)}{\partial \psi_2} - g_T(0, w) \frac{\partial p_T^*(2, w)}{\partial \psi_2} \right\} - \frac{\partial P(A_T(2))}{\partial \psi_2}, \\
\frac{\partial P(A_T(2))}{\partial \psi_2} &= -E_w \left[\frac{\partial q_T^*(w)}{\partial \psi_2} \{1 - g_T(0, w) - g_T(1, w) + p_T^*(1, w)g_T(0, w)\} \right], \\
\frac{\partial P(A_T(0) \cup A_T(1))}{\partial \psi_2} &= \frac{\partial P(A_T(0))}{\partial \psi_2} + \frac{\partial P(A_T(1))}{\partial \psi_2},
\end{aligned}$$

Numerical derivatives

Numerical derivatives are computed as local slopes based on small perturbations of parameter estimates (Press et al., 1992). Let $f(\omega)$ be the left hand side of (25), (26), or (32) where $\omega = (\psi_0, \psi_1, \psi_2, h_{T,1}(k), h_{T,2}(k))$, and let $\omega^{(-i)}$ be a vector with the i th element of ω removed. The i th element of ν , $\omega^{(i)}$, is perturbed as follows:

$$\begin{aligned}
\delta_i &= c_1 \left(\frac{\omega^{(i)}}{\omega^{(i)} + c_2} \right), \\
\omega_1^{(i)} &= \begin{cases} \omega^{(i)} - \delta_i, & \delta_i \geq c_2; \\ \omega^{(i)} - c_2, & \delta_i < c_2, \end{cases} \quad \omega_2^{(i)} = \begin{cases} \omega^{(i)} + \delta_i, & \delta_i \geq c_2; \\ \omega^{(i)} + c_2, & \delta_i < c_2, \end{cases}
\end{aligned}$$

where c_1 and c_2 are small (here, we set $c_1 = 0.01$ and $c_2 = 0.001$). Using the current estimates of the parameters ω , the derivative of $\omega^{(i)}$ is

$$\frac{\partial P(Y_T(\text{tx}) > k | A_T(0))}{\partial \omega^{(i)}} = \frac{f(\omega^{(-i)}, \omega_2^{(i)}) - f(\omega^{(-i)}, \omega_1^{(i)})}{2\delta_i},$$

for $\text{tx} = 0, 1, 2$.

References

Press, W. H., Teukolsky, S. A., Vetterling, W. T., and Flannery, B. P. (1992). *Numerical Recipes in C: The Art of Scientific Computing*. Cambridge University Press, Cambridge, UK, second edition.