HW 6 sol'ns

1.

Let $m_i = y_{1i} + y_{2i}$, $(i = 1, \cdots, n)$ and $\Delta = log\frac{\pi_{1i}/(1-\pi_{1i})}{\pi_{2i}(1-\pi_{2i})}$. Then

$$p(y_{1i} = 1|m_i = 1) = \frac{p(y_{1i} = 1, m_i = 1)}{p(m_i = 1)} = \frac{p(y_{1i} = 1, y_{1i} + y_{2i} = 1)}{p(y_{1i} + y_{2i} = 1)}$$

$$= \frac{p(y_{1i} = 1, y_{2i} = 0)}{p(y_{1i} = 1, y_{2i} = 0) + p(y_{1i} = 0, y_{2i} = 1)}$$

$$= \frac{\pi_{1i}(1-\pi_{2i})}{\pi_{1i}(1-\pi_{2i}) + (1-\pi_{1i})\pi_{2i}} = \frac{\frac{\pi_{1i}(1-\pi_{2i})}{\pi_{2i}(1-\pi_{1i})}}{1 + \frac{\pi_{1i}(1-\pi_{2i})}{\pi_{2i}(1-\pi_{1i})}}$$

$$= \frac{e^\Delta}{1+e^\Delta}$$

We have the conditional likelihood

$$l_c(\Delta) = \sum_{i=1}^n log\left[\left(\frac{e^\Delta}{1+e^\Delta}\right)^{y_{1i}}\left(\frac{1}{1+e^\Delta}\right)^{1-y_{1i}}\right]$$

$$= \sum_{i=1}^n \left[log\left(e^{\Delta y_{1i}}\right) - log\left(1+e^\Delta\right)\right]$$

$$= \sum_{i=1}^n \Delta y_{1i} - nlog(1+e^\Delta)$$

$$= \Delta y_{1\cdot} - nlog(1+e^\Delta)$$

Thus, the score statistic for the null hypothesis $H_0 : \Delta = 0$ is

$$U = \frac{\partial l_c}{\partial \Delta}|_{\Delta=0} = \left(y_{1\cdot} - n\frac{e^\Delta}{1+e^\Delta}\right)|_{\Delta=0} = y_{1\cdot} - \frac{n}{2}$$

2.

Profile log likelihood for ψ is denoted as $l^p(\psi, y) = l(\psi, \hat{\lambda}_\psi, y)$. By Taylor expansion, we obtain

$$\begin{aligned}
\frac{\partial l^p}{\partial \psi} &= \frac{\partial l}{\partial \psi} + \frac{\partial^2 l}{\partial \psi \partial \lambda}(\hat{\lambda}_\psi - \lambda) + \frac{1}{2}\frac{\partial^3 l}{\partial \psi \partial \lambda^2}(\hat{\lambda}_\psi - \lambda)^2 + \cdots \\
&\quad + \left\{ \frac{\partial l}{\partial \lambda} + \frac{\partial^2 l}{\partial \lambda^2}(\hat{\lambda}_\psi - \lambda) + \frac{1}{2}\frac{\partial^3 l}{\partial \lambda^3}(\hat{\lambda}_\psi - \lambda)^2 + \cdots \right\} \frac{\partial \hat{\lambda}_\psi}{\partial \psi} \\
&= \frac{\partial l}{\partial \psi} + \frac{\partial^2 l}{\partial \psi \partial \lambda}(\hat{\lambda}_\psi - \lambda) + \frac{1}{2}\frac{\partial^3 l}{\partial \psi \partial \lambda^2}(\hat{\lambda}_\psi - \lambda)^2 + \cdots \\
&= O_p(n^{\frac{1}{2}}) + O_p(n^{\frac{1}{2}}) + O_p(1) + \cdots \quad \text{(for large n)}
\end{aligned}$$

because

$$\frac{\partial l(\psi, \lambda)}{\partial \lambda}|_{\hat{\lambda}_\psi = 0} = \frac{\partial l}{\partial \lambda} + \frac{\partial^2 l}{\partial \lambda^2}(\hat{\lambda}_\psi - \lambda) + \frac{1}{2}\frac{\partial^3 l}{\partial \lambda^3}(\hat{\lambda}_\psi - \lambda)^2 + \cdots = 0$$

Hence,

$$\begin{aligned}
E\left[\frac{\partial l^p}{\partial \psi}\right] &= E\left[\frac{\partial l}{\partial \psi}\right] + E\left[\frac{\partial^2 l}{\partial \psi \partial \lambda}(\hat{\lambda}_\psi - \lambda)\right] + \frac{1}{2}E\left[\frac{\partial^3 l}{\partial \psi \partial \lambda^2}(\hat{\lambda}_\psi - \lambda)^2\right] + \cdots \\
&= 0 + O(1) + O(1) + \cdots \neq 0
\end{aligned}$$

If $\hat{\lambda}_\psi$ is not consistent estimate of λ, these expectations may be very large. Thus $E\left[\frac{\partial l^p}{\partial \psi}\right]$