Consider the standard situation for hypothesis testing that we have assumed in the previous lectures. We have data from a population with an unknown parameter $\theta$. Let us assume we have an estimator $\hat{\theta}$ of $\theta$ such that

$$\frac{\hat{\theta} - \theta}{\hat{SE}(\theta)}$$

is approximately Normal$(0,1)$.

Suppose we want to test the two hypotheses

$$H_0: \theta = \theta_0 \quad \text{v.s.} \quad H_a: \theta \neq \theta_0.$$ 

We derived the following testing procedure with level $\alpha$.

**TEST STATISTIC:** $\frac{\hat{\theta}}{\hat{SE}(\theta)}$

**REJECTION REGION:** $\hat{\theta} < \theta_0 + z_{\alpha/2} \hat{SE}(\theta)$ or $\hat{\theta} > \theta_0 + z_{1-\alpha/2} \hat{SE}(\theta)$.
Alternatively, we accept \( H_0 \) if
\[
0_0 + z_{1-\frac{\alpha}{2}} \hat{SE}(\theta) < \hat{\theta} < 0_0 + z_{1-\frac{\alpha}{2}} \hat{SE}(\theta)
\]

\[\iff\]
\[
\hat{\theta} - z_{1-\frac{\alpha}{2}} \hat{SE}(\theta) < 0_0 < \hat{\theta} + z_{1-\frac{\alpha}{2}} \hat{SE}(\theta)
\]

\[\iff\]
\[
0_0 \in \left[ \hat{\theta} - z_{1-\frac{\alpha}{2}} \hat{SE}(\theta), \hat{\theta} + z_{1-\frac{\alpha}{2}} \hat{SE}(\theta) \right].
\]

Now, let us consider the problem of deriving a \((1-\alpha)\) confidence interval for \( \theta \). Since
\[
g(\text{data, } \theta) = \frac{\hat{\theta} - \theta}{\hat{SE}(\theta)} \text{ is approximately Normal}(0, 1),
\]
it can be used as a pivotal quantity.

**Step 1:** Use \( \frac{\hat{\theta} - \theta}{\hat{SE}(\theta)} \) as a pivotal quantity.

**Step 2:** Find \( a, b \) such that
\[
P\left( a \leq \frac{\hat{\theta} - \theta}{\hat{SE}(\theta)} \leq b \right) = 1-\alpha.
\]
Note that

\[ P \left( -z_{1-\frac{\alpha}{2}} \leq \text{Normal}(\mu) \leq z_{1-\frac{\alpha}{2}} \right) = 1-\alpha \]

It follows that \( a = -z_{1-\frac{\alpha}{2}} \) and \( b = z_{1-\frac{\alpha}{2}} \).

**Step 3:** Convert \( a \leq \frac{\hat{\theta} - \theta}{\hat{SE}(\theta)} \leq b \) to the form \( -\frac{1}{b} \leq \frac{\hat{\theta} - \theta}{\hat{SE}(\theta)} \leq \frac{1}{a} \).

Note that

\[-z_{1-\frac{1}{\alpha}} \leq \frac{\hat{\theta} - \theta}{\hat{SE}(\theta)} \leq z_{1-\frac{1}{\alpha}}\]

\[\iff \hat{\theta} - z_{1-\frac{1}{\alpha}} \hat{SE}(\theta) \leq \theta \leq \hat{\theta} + z_{1-\frac{1}{\alpha}} \hat{SE}(\theta)\]

\[\iff \theta \in \left[ \hat{\theta} - z_{1-\frac{1}{\alpha}} \hat{SE}(\theta), \hat{\theta} + z_{1-\frac{1}{\alpha}} \hat{SE}(\theta) \right] \quad - (2)\]
Hence, comparing (1) and (2), we obtain that the testing procedure for $H_0: \theta = \theta_0$ v.s. $H_1: \theta \neq \theta_0$ accepts $H_0$ if $\theta_0$ lies in the $(1-\alpha)$-confidence interval for $\theta$.

We established above a connection between the hypothesis testing procedure for a two-sided alternative hypothesis $H_0: \theta = \theta_0$ and the two-sided confidence interval for $\theta$. A similar connection can be established between one-sided alternative hypotheses and corresponding one-sided confidence intervals. See Homework 7, Problem 10.46.