LECTURE – 10

Agenda:

1. Expected values
2. Variance
3. Properties of expected values and variance

EXPECTED VALUES

Suppose we are interested in a random variable X arising out of a random experiment. Based on our understanding of the random experiment, we have a probability model. Often, we want to summarize our understanding of the random variable in one number, "The expected value" of the random variable.

DEFINITION: The "expected value" of a discrete random variable X with probability mass function p_X is given by

$$E(X) = \sum_{x \in \mathcal{X}} x\, p_X(x) = \sum_{x \in \mathcal{X}} x\, P(X=x).$$

It is also understood as our estimate of the "average" value that the random variable will take.

NOTE: The expected value of a discrete random variable is defined only if $\sum_{x \in \mathcal{X}} |x|\, P(X=x) < \infty$.

Example: Consider the following game. We toss a six-faced die 2 times. If the sum of the two values is 3 or lower, we have to pay 10 dollars. If the sum of the two values is 4, 5 or 6 we pay 4 dollars. If the sum of the two values is 7, 8 or 9 we earn 4 dollars. If the sum of the two values is 10, 11 or 12 we earn 10 dollars. What are the expected winnings?

Let X = Sum of two values on the die

$$P(X = 2 \text{ or } 3) = \frac{3}{36} = \frac{1}{12}$$

$$P(X = 4, 5 \text{ or } 6) = \frac{12}{36} = \frac{1}{3}$$

$$P(X = 7, 8 \text{ or } 9) = \frac{15}{36} = \frac{5}{12}$$

$$P(X = 10, 11 \text{ or } 12) = \frac{6}{36} = \frac{1}{6}$$

Let W = Winnings in the game

$P(W = -10) = \frac{1}{12}$, $P(W = -4) = \frac{1}{3}$, $P(W = 4) = \frac{5}{12}$,

$P(W = 10) = \frac{1}{6}$

$$E[W] = \left(-10 \times \frac{1}{12}\right) + \left(-4 \times \frac{1}{3}\right) + \left(4 \times \frac{5}{12}\right) + \left(10 \times \frac{1}{6}\right)$$

$$= \frac{-10 - 16 + 20 + 20}{12}$$

$$= \frac{7}{6}$$

RESULT: If X is a discrete random variable with probability distribution p_X and if $g: \mathbb{R} \to \mathbb{R}$ is a real-valued function, then

$$E(g(X)) = \sum_{x \in \mathcal{X}} g(x)\, p_X(x),$$

If $g(x) = x^2$, then $E[g(W)]$ is given by

$$E[W^2] = 100 \times \frac{1}{12} + 16 \times \frac{1}{3} + 16 \times \frac{5}{12} + 100 \times \frac{1}{6}$$

$$= 25 + 12$$

$$= 37.$$

VARIANCE

DEFINITION: The variance of a random variable X with expected value μ is given by

$$V(X) = E(X-\mu)^2 = \sum_{x \in \mathcal{X}} (x-\mu)^2 p_X(x).$$

PROPERTIES OF EXPECTATION AND VARIANCE

Note that the variance $V(X)$ of a random variable is the average squared distance between the values of X and the expected value μ.

DEFINITION: The <u>standard deviation</u> of a random variable X is the square root of the variance and is given by

$$SD(X) = \sqrt{E(X-\mu)^2}$$

The standard deviation is also a measure of the variability of a random variable, but it maintains the original units of measure. It can be thought of as the size of a typical deviation between an observed outcome and the expected value.

Example: If W is the winnings in the game discussed in the previous lecture, remember

$P(W = -20) = \frac{1}{12}$, $P(W=-4) = \frac{1}{3}$, $P(W=4) = \frac{5}{12}$,

$P(W = 10) = \frac{1}{6}$.

$\boxed{(ii)}$ $V(aX+b) = a^2 V(X)$

Proof:

$$V(aX+b) = E\left[(aX+b) - E(aX+b)\right]^2$$

$$= E\left[(aX+b) - aE(X) - b\right]^2$$

$$= E\left[a(X - E(X))\right]^2$$

$$= a^2 E\left[(X - E(X))^2\right]$$

$$= a^2 V(X)$$

$\boxed{(iii)}$ $V(X) = E(X^2) - (E(X))^2$

$$V(X) = E(X-\mu)^2$$

$$= E(X^2 - 2X\mu + \mu^2)$$

$$= E(X^2) - 2\mu E(X) + E(\mu^2)$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2$$

RESULT: Let X be a random variable with mean μ and variance σ^2. Then for any positive k,

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

Proof: Let $V(X) = \sigma^2$ and $E(X) = \mu$

$$V(X) = \sum_{x \in \mathcal{X}} (x-\mu)^2 p_X(x)$$

$$= \sum_{x \in \mathcal{X} : |x-\mu| \geq k\sigma} (x-\mu)^2 p_X(x) + \sum_{x \in \mathcal{X} : |x-\mu| < k\sigma} (x-\mu)^2 p_X(x)$$

$$\geq \sum_{x \in \mathcal{X} : |x-\mu| \geq k\sigma} (x-\mu)^2 p_X(x)$$

$$\geq k^2\sigma^2 \sum_{x \in \mathcal{X} : |x-\mu| \geq k\sigma} p_X(x)$$

$$= k^2\sigma^2 P(|X-\mu| \geq k\sigma)$$

Hence, $\sigma^2 \geq k^2\sigma^2 P(|X-\mu| \geq k\sigma)$

Hence, $\frac{1}{k^2} \geq P(|X-\mu| \geq k\sigma)$

This gives us the required identity as

$$\frac{1}{k^2} \geq 1 - P(|X-\mu| < k\sigma)$$

$$\Rightarrow P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}.$$