PhD Qualifying Examination
Department of Statistics, University of Florida
January 24, 2003, 8:00 am - 12:00 noon

Instructions:

1. You have exactly four hours to answer questions in this examination.
2. There are 8 problems of which you must answer 6.
3. Only your first 6 problems will be graded.
4. Write only on one side of the paper, and start each question on a new page.
5. Write your **number** on every page.
6. Do not write your name anywhere on your exam.
7. You must show your work to receive credit.
8. While the eight questions are equally weighted, within a given question, the parts may have different weights.

The following abbreviations are used throughout:

- GLM = generalized linear model
- mgf = moment generating function
- UMP = uniformly most powerful

1. Let $\{X_n, n \geq 1\}$ be a sequence of random variables and let $S_n = \sum_{j=1}^{n} X_j$, $n \geq 1$. Prove that if $\sum_{n=1}^{\infty} \mathrm{E}\,|X_n| < \infty$, then there exists a random variable S with $S_n \to S$ almost certainly and $S_n \xrightarrow{\mathcal{L}_1} S$.

2. Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with $\mathrm{E}\,|X_1|^q < \infty$ for some $q \in (0, \infty)$. Prove that for all $p \in (1, \infty)$,

$$\frac{\sum_{j=1}^{n} |X_j|^{pq}}{n^p} \to 0 \quad \text{almost certainly.}$$

3. (a) Suppose that $L \sim \text{Poisson}(\phi)$ and that $Y|L \sim \chi^2_{q+2L}$; that is, conditional on L, Y has a χ^2 distribution with $q+2L$ degrees of freedom. Write down the marginal density function of Y. You should recognize this as the *non-central* χ^2 distribution with q degrees of freedom and non-centrality parameter ϕ.

(b) Find the mean of Y.

(c) Let $X_1, \ldots, X_p$ be independent random variables such that $X_i \sim \mathrm{N}(\theta_i, 1)$ for $i = 1, \ldots, p$. Assume that $p > 2$. Put $X = (X_1, \ldots, X_p)^T$, $\theta = (\theta_1, \ldots, \theta_p)^T$ and $\lambda = \|\theta\|^2/2$. Show that

$$\mathrm{E}\left(\frac{1}{\|X\|^2}\right) = \mathrm{E}[g(K)] \tag{1}$$

where $K \sim \text{Poisson}(\lambda)$. In other words, identify the function g. (In order to answer this question, you need to know the distribution of $\|X\|^2$. However, you are *not required* to derive this distribution.)

(d) The equation (1) can clearly be rewritten as

$$\int_{\mathbb{R}^p} \frac{1}{\|x\|^2 (2\pi)^{p/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{p}(x_i - \theta_i)^2\right\} dx = \sum_{k=0}^{\infty} g(k) \frac{\exp\{-\lambda\}\,\lambda^k}{k!} \ .$$

Assuming that $\frac{\partial}{\partial \theta_j}$ can be passed through the integral and through the sum, differentiate both sides to show that

$$\mathrm{E}\left(\frac{X_j}{\|X\|^2}\right) = \frac{\theta_j}{\lambda}\mathrm{E}\left(\frac{K}{p-2+2K}\right) \ .$$

(e) Consider the James-Stein estimator of θ given by

$$\delta(X) = \left(1 - \frac{p-2}{\|X\|^2}\right) X \ .$$

Use the above results to show that the mean squared error of δ can be written as

$$\mathrm{E}\,\|\,\delta(X) - \theta\,\|^2 = p - (p-2)^2 \mathrm{E}\left(\frac{1}{p-2+2K}\right) \ .$$

(f) In the context of estimating θ under squared error loss, what have we shown?

4. (a) Suppose $Z|\theta \sim$ Geometric(θ); that is,

$$P(Z = z|\theta) = \theta(1-\theta)^z$$

for $z \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and $\theta \in (0, 1)$. Note that $\mathrm{E}(Z|\theta) = \frac{1-\theta}{\theta}$ and $\mathrm{Var}(Z|\theta) = \frac{1-\theta}{\theta^2}$. Find the marginal mass function of Z assuming that $\theta \sim$ Beta(α, β).

(b) The function $\psi(x) = \frac{d}{dx} \log \Gamma(x)$ (defined for positive x) is called the *digamma function*. The digamma function has the following integral representation

$$\psi(x) = -\gamma + \int_0^1 \frac{1 - t^{x-1}}{1-t}\, dt \,,$$

where γ is *Euler's constant*. Use this representation to show that $\psi(x)$ is an increasing function. (Hint: You don't need any derivatives.)

(c) Now use the fact that ψ is increasing to show that for fixed $0 < a < b$, the function

$$g(t) = \frac{\Gamma(t+a)}{\Gamma(t+b)}$$

is decreasing in t. (Hint: Use a log and a derivative.)

(d) Suppose we have a single observation from the mass function

$$P_\alpha(Z = z) = \frac{\alpha^2\, \Gamma(\alpha)\, z!}{\Gamma(z + \alpha + 2)}$$

for $z \in \mathbb{Z}_+$. Construct a UMP size 0.10 test of $H_0 : \alpha \le 3$ versus $H_A : \alpha > 3$. (Hint: You are *not* being forced to use the Neyman-Pearson Lemma here.)

5. Let $\boldsymbol{Y} = (Y_1, \dots, Y_k)$ be a multinomial vector of counts based on m trials and probability vector $\boldsymbol{\pi} = (\pi_1, \dots, \pi_k)$.

(a) Show that the joint mgf of $\boldsymbol{Y}$ is

$$M_{\boldsymbol{Y}}(\boldsymbol{t}) = \left(\sum_{j=1}^{k} \pi_j e^{t_j} \right)^m .$$

HINT: Use the identity,

$$\left(\sum_{j=1}^{k} \alpha_j \right)^m = \sum_{\boldsymbol{y} \in S} \frac{m!}{\prod_j y_j!} \prod_j \alpha_j^{y_j} \,,$$

where $S = \{\boldsymbol{y} = (y_1, \dots, y_k) | y_j \ge 0,\ \sum y_j = m\}$.

(b) Derive the mean vector and covariance matrix of $\boldsymbol{Y}$.

(c) Let $\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_n$ be independent multinomial vectors each with k categories. Suppose that $\boldsymbol{Y}_i$ is based on m_i trials and probability vector $\boldsymbol{\pi}_i = (\pi_{i1}, \dots, \pi_{ik})$, $i = 1, \dots, n$. Suppose further that the π_{ij}'s satisfy the model,

$$\pi_{ij} = \frac{\exp(\boldsymbol{x}_{ij}'\boldsymbol{\beta})}{\sum_{r=1}^{k} \exp(\boldsymbol{x}_{ir}'\boldsymbol{\beta})} \,,$$

where $\boldsymbol{x}_{ij}$ is a vector of known covariates associated with the (i, j)th count. Write down the log-likelihood function for the parameter $\boldsymbol{\beta}$. Show that there exists a Poisson loglinear GLM for which likelihood inference concerning $\boldsymbol{\beta}$ is identical to that based on this multinomial model.

6. Suppose that Y has a binomial distribution with m trials and probability π.

(a) Express the binomial likelihood function in exponential form in terms of the canonical parameter $\theta = \text{logit}(\pi)$.

(b) Derive the deviance measure of fit $D(y, \mu)$ for the binomial model, where $\mu = m\pi$.

(c) Show that the deviance can be approximated by the Pearson χ^2 statistic, X^2, if m is large, where

$$X^2 = \frac{m(p-\pi)^2}{\pi(1-\pi)},$$

and $p = Y/m$.

(d) Argue that, for $c > 0$

$$\text{E}\{\log(Y + c)\} = \log(m\pi) + \frac{c}{m\pi} - \frac{1-\pi}{2m\pi} + O(m^{-3/2}).$$

Hence show that

$$\text{E}\left\{\log\left(\frac{Y+c}{m-Y+c}\right)\right\} = \theta + \frac{(1-2\pi)(c-\frac{1}{2})}{m\pi(1-\pi)} + O(m^{-3/2}).$$

(e) Comment briefly on the relevance of the result in (d).

7. Let $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ be a quadratic form in $\boldsymbol{x}$ which is distributed as $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

(a) Give a complete expression for $\phi(t)$, the mgf of $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$.

(b) Show that $\phi(t)$ exists if $|t| < c$ for some constant c (specify what c is).

(c) Make use of (a) to show that if $\boldsymbol{A}\boldsymbol{\Sigma}$ is idempotent of rank r, then $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ is distributed as $\chi'^2_r(\lambda)$. Please specify what the non-centrality parameter is.

8. Consider the model $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\boldsymbol{X}$ is $n \times p$ of rank $r(< p)$, $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_n)$. Let M be an s-dimensional subspace of the row space of $\boldsymbol{X}(s \leq r)$ and let $\boldsymbol{C}'$ be a matrix of order $s \times p$ and rank s whose rows form a basis for M. It is known that Scheffé's simultaneous $(1-\alpha)100\%$ confidence intervals on all estimable linear functions of the form $\boldsymbol{a}'\boldsymbol{\beta}$, where $\boldsymbol{a}' \in M$, are given by

$$\boldsymbol{a}'\hat{\boldsymbol{\beta}} \pm \left\{ s[\boldsymbol{a}'(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{a}] MS_E F_{\alpha,s,n-r} \right\}^{1/2}, \tag{2}$$

where MS_E is the error mean square and $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y}$. (You do not have to prove (2)).

(a) The F-test concerning the hypothesis $H_0 : \boldsymbol{C}'\boldsymbol{\beta} = \mathbf{0}$ is significant at the α-level if and only if there exists $\boldsymbol{a}_0' \in M$ such that

$$|\boldsymbol{a}_0'\hat{\boldsymbol{\beta}}| > \{ s[\boldsymbol{a}_0'(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{a}_0] MS_E \, F_{\alpha,s,n-r} \}^{1/2}. \tag{3}$$

(b) Write $\boldsymbol{a}_0'$ in inequality (3) as $\boldsymbol{a}_0' = \boldsymbol{b}_0'\boldsymbol{C}'$, where $\boldsymbol{b}_0$ is some vector in $\mathbb{R}^s$, the s-dimensional Euclidean space. Show that inequality (3) is equivalent to

$$\sup_{\boldsymbol{b} \in \mathbb{R}^s} \frac{|\boldsymbol{b}'\boldsymbol{C}'\hat{\boldsymbol{\beta}}|}{|\boldsymbol{b}'\boldsymbol{C}'(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}\boldsymbol{b}|^{1/2}} > (sMS_E \, F_{\alpha,s,n-r})^{1/2} \tag{4}$$

(c) Show that inequality (4) can be written as

$$\sup_{\boldsymbol{b} \in \mathbb{R}^s} \frac{\boldsymbol{b}'\boldsymbol{G}_1\boldsymbol{b}}{\boldsymbol{b}'\boldsymbol{G}_2\boldsymbol{b}} > s \, MS_E \, F_{\alpha,s,n-r},$$

where $\boldsymbol{G}_1 = \boldsymbol{C}'\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}'\boldsymbol{C}$, $\boldsymbol{G}_2 = \boldsymbol{C}'(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}$.

(d) Show that

$$\sup_{\boldsymbol{b} \in R^s} \frac{\boldsymbol{b}'\boldsymbol{G}_1\boldsymbol{b}}{\boldsymbol{b}'\boldsymbol{G}_2\boldsymbol{b}} = e_{\max}(\boldsymbol{G}_2^{-1}\boldsymbol{G}_1),$$

where $e_{\max}(\boldsymbol{G}_2^{-1}\boldsymbol{G}_1)$ is the largest eigenvalue of $\boldsymbol{G}_2^{-1}\boldsymbol{G}_1$.

(e) Show that

$$\frac{\boldsymbol{b}'\boldsymbol{G}_1\boldsymbol{b}}{\boldsymbol{b}'\boldsymbol{G}_2\boldsymbol{b}}$$

attains its supremum if $\boldsymbol{b}$ is an eigenvector of $\boldsymbol{G}_2^{-1}\boldsymbol{G}_1$ corresponding to $e_{\max}(\boldsymbol{G}_2^{-1}\boldsymbol{G}_1)$.